Problem Set 3

This problem set is due Wednesday, October 5 at noon.

1. Let $P := \{p_1, \ldots, p_k\}$ be points in the plane and $\{Q_1, \ldots, Q_t\}$ be a partition of P into t sets. Argue (informally) that there exist disjoint curves in the plane, $C_1, \ldots, C_t$, such that for $i = 1, \ldots, t$, $Q_i \subseteq C_i$.

   Deduce (informally) that there is a suitable function g, such that for every k, the planar grid graph G of size $g(k) \times g(k)$ has the following property: if $u_1, \ldots, u_k \in V(G)$ are sufficiently far apart from each other and from the boundary (i.e. their pairwise distance and distance to the boundary is at least $f(k)$ for a suitable function f) and $\{Q_1, \ldots, Q_t\}$ is a partition of $\{u_1, \ldots, u_k\}$, then there exist disjoint trees $T_1, \ldots, T_t$ in G such that for $i = 1, \ldots, t$, $Q_i \subseteq T_i$.

   Solution: Let Π be the plane and observe that if F is a finite set of points in Π then $\Pi - F$ is still connected. So for any two given points $p, q \in \Pi - F$, we can find a simple curve C in $\Pi - F$ that connects p and q. But $\Pi - F - C$ is still connected because it is homeomorphic to $\Pi - F'$, where $F' := F \cup \{p\}$ is a finite set of points. So we can proceed inductively and find all the required curves that connect $Q_1, \ldots, Q_t$.

   Now observe that a grid is essentially a discrete plane. If the grid is fine enough and the given points are distant enough from each other and the boundary, we can discretize the curves that we got in the previous part so that they are paths on the grid.

2. For a vertex v in a graph G and a permutation π_v of its neighbors, define the operation $\texttt{split}(v, \pi_v)$ as replacing v by a path P_v of length degree(v) and connecting each of the neighbors of v to exactly one vertex of P_v in the order specified by π_v.

   Show that for every given integer $k \geq 1$, there exists a planar grid graph augmented by an apex vertex v (i.e. a vertex that is allowed to be connected to every other vertex) such that for any permutation π_v, the graph obtained from applying $\texttt{split}(v, \pi_v)$ contains K_k as a minor.

   Solution:

   For a given k, consider the planar grid graph of size $g(k^2) \times g(k^2)$ where g is the function from Problem 1. Let $u_1, \ldots, u_{k^2}$ be vertices on the grid that are at least $f(k^2)$ away from each other and the boundary, where f is the function from Problem 1. Define G as this grid graph augmented by an apex vertex v that is connected to $u_1, \ldots, u_{k^2}$.

   Applying $\texttt{split}(v, \pi_v)$ to obtain a path P_v imposes a linear order on the vertices $\{u_1, \ldots, u_{k^2}\}$. W.l.o.g. assume that this order is $u_1, u_2, \ldots, u_{k^2}$. Let σ be a sequence of length k^2 comprised of the numbers $1, \ldots, k$ such that each pair $1 \leq i < j \leq k$

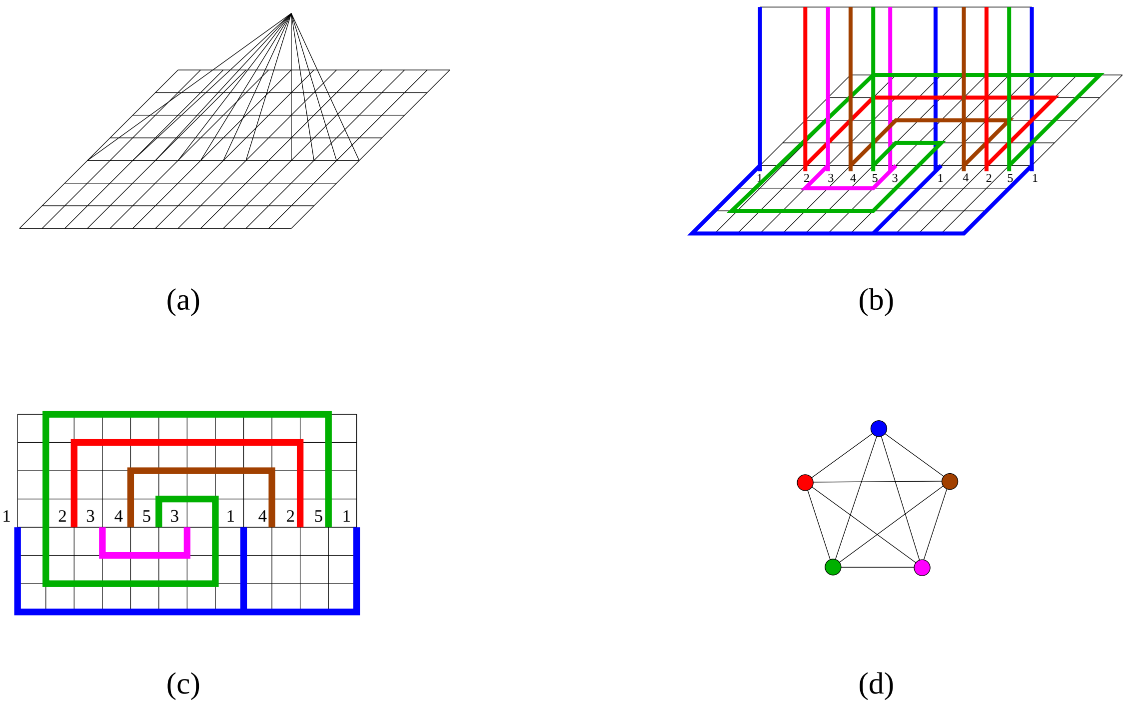

Figure 1: The apex of the graph in (a) is split, resulting in the graph (b); the vertices are labeled and connected according to the disjoint trees in (c); contracting the thick edges in (b) results in a K_5-minor (d).

is adjacent at least once. For $1 \leq i \leq k^2$, define the label of vertex u_i as $\sigma(i)$. For $1 \leq j \leq k$, let Q_j be the set of all vertices in $\{u_1, \ldots, u_{k^2}\}$ with label j and let T_j be a tree that connects all the vertices of Q_j on the grid in such a way that $T_1, \ldots, T_k$ are disjoint. These trees exist by Problem 1. Now contracting each tree and the edges that connect $u_1, \ldots, u_{k^2}$ to P_v results in a K_k-minor (the edges of P_v are the edges of the K_k-minor).

Note: Note that applying the `split` operation to every vertex results in a graph of degree at most 3 and note that a grid graph augmented by an apex – a so-called apex-graph – is K_6-minor-free. Hence, there are K_6-minor-free graphs where "no matter how" we split their vertices to reduce to a bounded-degree instance, we will introduce arbitrarily large minors.