Problem Set 10 - Solutions

In this problem set you will develop an algorithm for canceling flow cycles in a given flow assignment. In general graphs this can be done in $O(m \log n)$ time using Sleator’s and Tarjan’s dynamic trees. You will use the relation between dual shortest paths and circulations to give a linear time algorithm in planar graphs.

1. Let $G$ be a planar graph with non-negative capacities on its arcs. Let $\phi$ be shortest path distances from $f_\infty$ in $G^*$. Let $\theta$ be the circulation induced by $\phi$. That is,

$$\theta(d) = \phi(\text{head}(d)) - \phi(\text{tail}(d)).$$

Recall that a residual path is a path whose darts all have strictly positive capacities. Show that there are no counterclockwise residual cycles in the residual graph $G_\theta$.

**Solution:** Let $T^*$ be a shortest path tree in $G^*$ rooted at $f_\infty$, and let $C$ be a counterclockwise cycle. Since $C$ encloses at least one face (and by definition does not enclose $f_\infty$), and since $T^*$ is a spanning tree of $G^*$, there must be a dart $d^*$ of $T^*$ whose dual $d$ belongs to $C$. By definition of $\theta$, the residual capacity of darts of $T^*$ is zero, so $C$ is not residual.

2. What price function $\phi'$ would you use to get the same property as in (1), but with no clockwise residual cycles?

**Solution:** Reverse the directions of all darts in $G^*$. Then the argument applies to clockwise cycles in $G$.

3. Use parts (1) and (2) to give a linear time algorithm that, given a flow assignment $\gamma$ in $G$ makes $\gamma$ acyclic by removing all flow cycles in $\gamma$. That is, it produces another flow assignment $\gamma'$ s.t.

   (a) $\gamma' - \gamma$ is a circulation
   (b) for every arc $a$, $\gamma'((a,1)) \leq \gamma((a,1))$
   (c) for any cycle $C$ there is a dart $d \in C$ s.t. $\gamma'(d) \leq 0$

**Solution:** Consider the graph $G$ with arc capacities $\gamma(a)$. Let $\theta_1$ be the circulation from part 1. Let $\gamma_1$ be the flow obtained from $\gamma$ by pushing $-\theta_1$ (i.e., $\gamma_1 = \gamma - \theta_1$). $\gamma_1$ consists of no counterclockwise cycles, and $\gamma_1((a,1)) \leq \gamma((a,1))$ for all arcs $a$.

Next consider the graph $G$ with arc capacities $\gamma_1$. Let $\theta_2$ be the circulation from part 2. Let $\gamma_2$ be the flow obtained from $\gamma_1$ by pushing $-\theta_2$ (i.e., $\gamma_1 = \gamma - \theta_1 - \theta_2$). $\gamma_2$ consists of no clockwise cycles, and since $\gamma_2((a,1)) \leq \gamma_1((a,1))$ for all arcs $a$, no counterclockwise cycles are introduced. Hence $\gamma_2$ is the desired flow.