Recall interdigitating trees of planar graphs:

If $T$ is a spanning tree of the primal, then $E(G) - E(T)$ is a spanning tree $T^*$ of the dual.

On higher surfaces, we have the analogous concept of a tree-cotree decomposition:

A triple $(T, T^*, X)$ where:
\( T \) is a spanning tree of the primal, 
\( T^* \) is a spanning tree of the dual with \( E(T^*) \cap E(T) = \emptyset \)

\( X = E(G) - E(T) - E(T^*) \) is a set of exactly \( g \) edges, where \( g \) is the Euler genus (by Euler's formula).

Consider the graph obtained by pasting the faces of \( G \) only along edges of \( T^* \). Since \( T^* \) is a tree this is planar. In fact, it is the graph obtained by cutting \( G \) open along \( TVX \).

\( T \): 4 edges
\( T^* \): 4 edge
\( X \): 2 edges
Choose an arbitrary root for $T$.

For each edge $e \in X$ consider its fundamental cycle with $T$ and extend it to a closed walk that contains $r$.

We call this loop $(T, e)$.

Consider the system of loops through $r$ defined by

$$C = \{\text{loop}(T, e) : e \in X\}$$

The graph $G \cup E$ is called a cut graph for $G$ and is a subgraph of $G$ consisting of some $t$ vertices, $t+g-1$ edges, exactly 1 face, and having Euler genus $g$.

It is sufficient to cut $G$ open along $CG$ (instead of $T \cup X$) to obtain a planar graph. Indeed, we do not need to cut the "dangling" parts of $T$!

Note: compare fundamental polygon of surface.
Note: If we choose $T$ to be a shortest paths tree rooted at $v$, then $G$ consists of $2g$ shortest paths + $g$ additional edges.

Recall spanners (for opt. problems):

1. a subgraph of weight $O(\text{OPT})$
2. that contains a $(1+\varepsilon)$-approx. solution

If we make sure that every shortest path and every edge of the graph are of weight $O(\text{OPT})$, then $G$ is of weight $O(\text{OPT})$!

Spanner construction for Steiner tree in a bounded genus graph $G$:

1. Compute a 2-approximate Steiner tree $T_0$ and contract it to a single vertex $v$.
2. Compute a shortest-paths tree $\text{SPT}$ rooted at $v$.
3. Delete all vertices $v$ and all edges $e=uv$ with $d(v, u) > 2l(T_0)$, resp. $\min\{d(v, u), d(v, w) + l(e)\} > 2l(T_0)$.

Call this subgraph $G$. These vertices and edges can never be in an $(1+\varepsilon)$-approx. solution since they are more than $2\cdot\text{OPT}$ away from any terminal.

All shortest paths in $G$ are now at length at most $4\cdot\text{OPT}$. 

→ These vertices and edges can never be in an $(1+\varepsilon)$-approx. solution since they are more than $2\cdot\text{OPT}$ away from any terminal.

→ All shortest paths in $G$ are now at length at most $4\cdot\text{OPT}$.
1. Uncontract \( r \) and set \( T = T_0 \cup SPT \)
   \( T \) is a spanning tree of \( G \)

2. Find a spanning tree \( T^* \) in \( G^* - E(T) \)
   \( T^* \) is a spanning tree of \( G^* \)

3. Let \( X = E(G) - E(T) - E(T^*) \)
   \( (T, T^*, X) \) is a tree-cuttree decomposition

4. Pick an arbitrary root \( v \in V(T_0) \) for \( T \) and define
   \( CG = T_0 \cup \{ e \in E(T) : e \in X \} \)
   \( CG \) is a cut graph for \( G \) and
   \( \ell(CG) \leq 2\text{OPT} + \gamma \cdot 8\text{OPT} = (8\gamma + 2) \cdot \text{OPT} \)

5. Cut the graph open along \( CG \) to obtain a planar graph \( G_p \) with a distinguished face \( f \) that contains all terminals.
   \( \ell(f, f_{\text{outer}}) = 2 \cdot \ell(CG) \leq (16\gamma + 4) \cdot \text{OPT} \)

6. Proceed as in planar spanner construction!
   Find strips, columns, super-columns, subgraph, bisect
   portals, and add all possible Steiner trees
   among portals. \( \rightarrow \) see Lecture 16

   Crucial point: The bricks are planar
   and hence our planar structure theorem suffices!
PTAS for Steiner tree in bounded-genus graphs:

1. Find spanner as above
2. Apply contraction-decomposition to spanner
3. Solve on bounded treewidth
4. Uncontract the small part contracted in 2 to obtain solution for original graph — Lecture 23
Shallow minors:

We say $H$ is a depth-$r$ minor of $G$, $H \preceq_r G$, if $G$ contains a model of $H$ in which every branch-set has diameter at most $r$.

Hence, depth-$0$ minors are subgraphs
depth-$n$ minors are minors.

For a class of graphs $C$ define

$$C \mathcal{D}_r = \{ \text{the set of all depth-}$r$ minors of graphs of } C \}.$$

We say $C$ is somewhere dense if there exists an $r$ such that $C \mathcal{D}_r$ is equal to the set of all graphs (i.e., contains arbitrarily large cliques).
$C$ is nowhere dense if it is not somewhere dense, i.e. for every $r$, there exists a $k$ such that $C \notin r$ does not contain $K_k$.  

We say $C$ has bounded expansion if there is a function $f$, s.t. every graph in $C \notin r$ has average degree at most $f(r)$.  

First-order logic is FPT on bounded expansion.  

- is not FPT on somewhere dense  
- on nowhere dense: OPEN!  

However, subgraph isomorphism, dominating set, etc. are FPT on nowhere dense.  

In a nowhere dense class $C$ and given $k$, there exists a $t$ ($\gg k$) s.t. every graph in $C$ can be colored by $t$ colors, s.t. any $k$ colors induce a graph of bounded treewidth.
References


