6.889	Lecture 24	Dec. 7, 2011

Recall interdigitating trees of planar graphs:

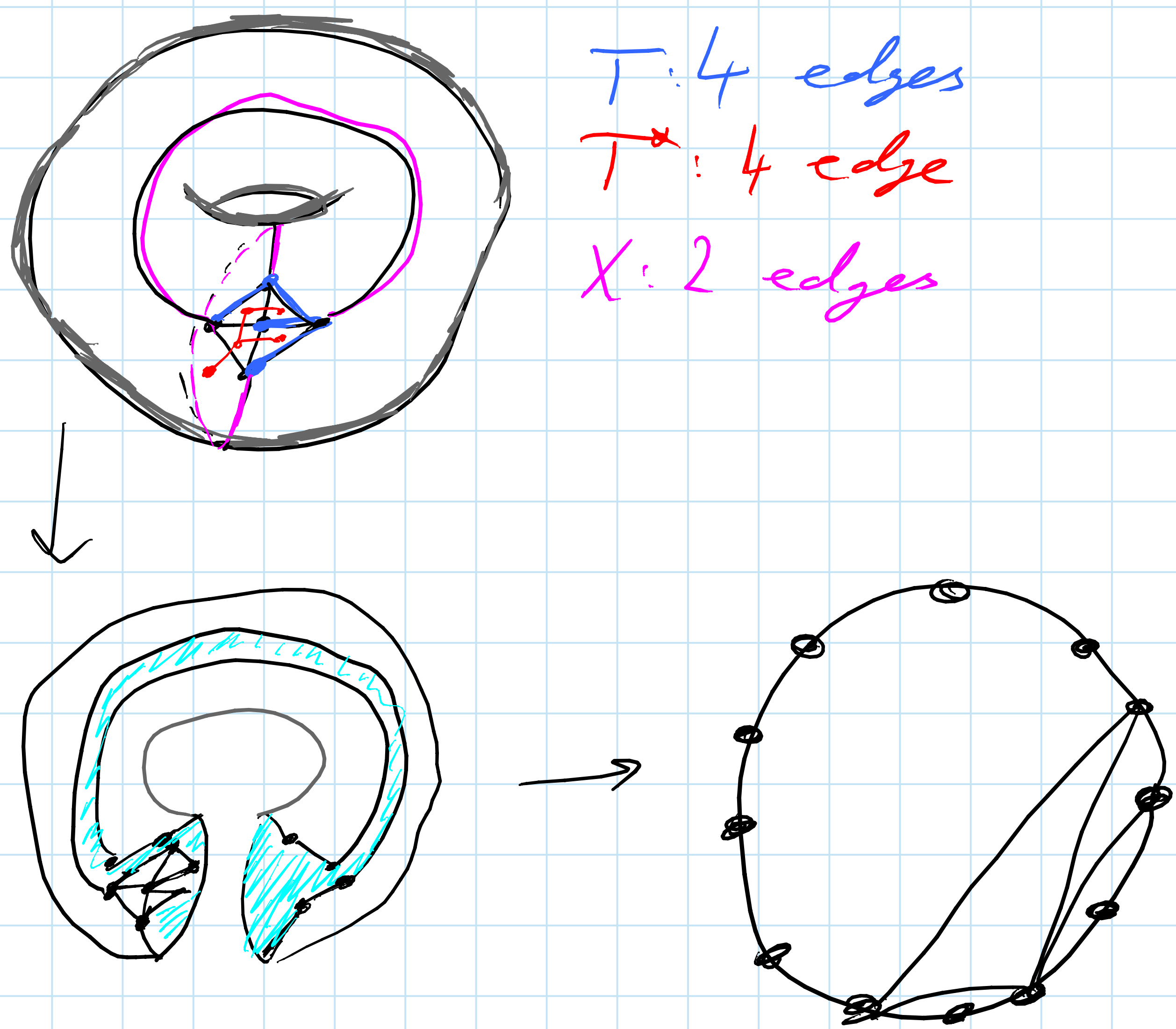
If T is a spanning tree of the primal, then $E(G) - E(T)$ is a spanning tree T^* of the dual.

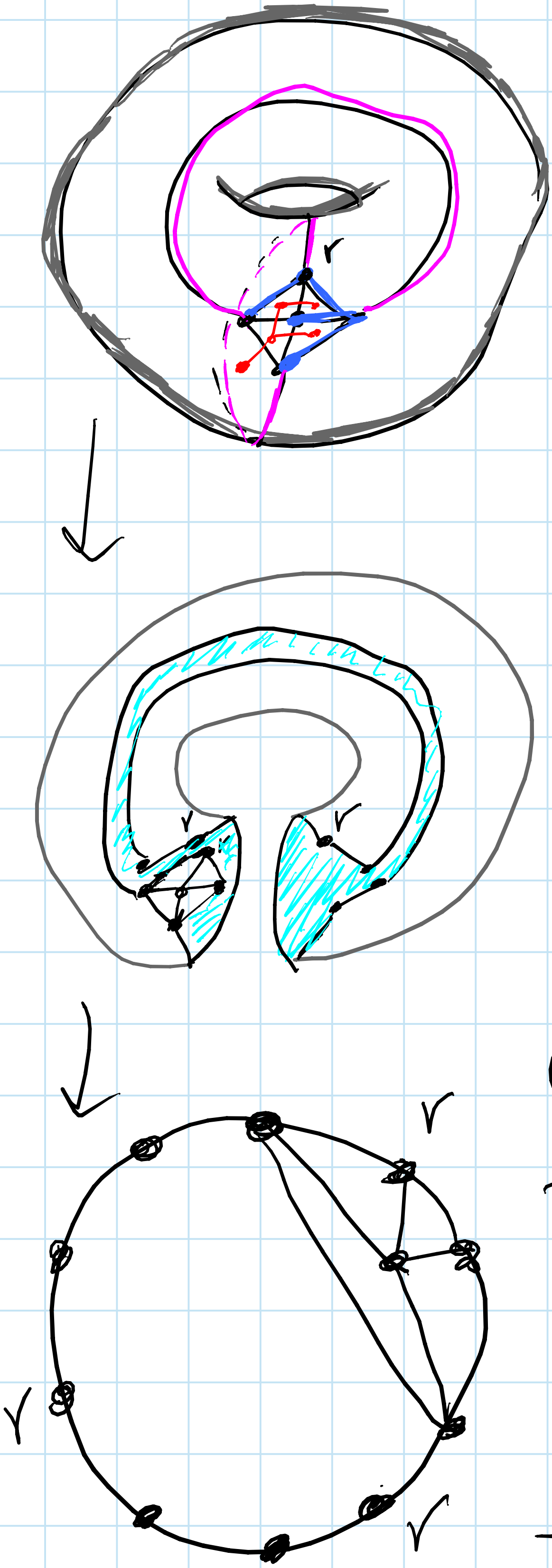
On higher surfaces, we have the analogous concept of a tree-cotree decomposition:

A triple (T, T^*, X) where:

T is a spanning tree of the primal,
T^* is a spanning tree of the dual with $E(T^*) \cap E(T) = \emptyset$,
$X = E(G) - E(T) - E(T^*)$ is a set of exactly g edges, where g is the Euler genus (by Euler's formula).

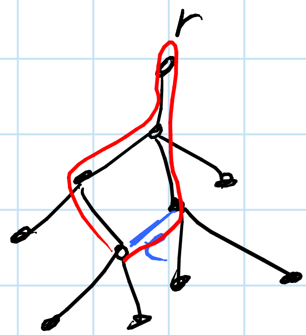
Consider the graph obtained by pasting the faces of G only along edges of T^*. Since T^* is a tree this is planar. In fact, it is the graph obtained by cutting G open along $T \cup X$.





Choose an arbitrary root r for T.

For each edge $e \in X$ consider its fundamental cycle with T and extend it to a closed walk that contains r:



We call this loop(T, e).

Consider the system of loops through r defined by

$$\mathcal{C} = \{ loop(T,e) : e \in X \}$$

The graph $CG = \bigcup \mathcal{C}$ is called a cut graph for G and is a subgraph of G consisting of some t vertices, $t+g-1$ edges, exactly 1 face, and having Euler genus g.

It is sufficient to cut G open along CG (instead of $T \cup X$) to obtain a planar graph. Indeed, we do not need to cut the "dangling" parts of T! ↳ compare fundamental polygon of surface.

Note: If we choose T to be a shortest paths tree rooted at r, then CG consists of $2g$ shortest paths $+ g$ additional edges.

Recall spanners (for opt. problems):

① a subgraph of weight $O(OPT)$
② that contains a $(1+\varepsilon)$-approx. solution

If we make sure that every shortest path and every edge of the graph are of weight $O(OPT)$, then CG is of weight $O(OPT)$!

Spanner construction for Steiner tree in a bounded genus graph G_0:

① Compute a 2-approximate Steiner tree T_0 and contract it to a single vertex r.
② Compute a shortest-paths tree SPT rooted at r.
③ Delete all vertices v and all edges $e = uw$ with $d(r, v) > 2\ell(T_0)$, resp. $\min\{d(r,u), d(r,w)\} + \ell(e) > 2\ell(T_0)$.

call this subgraph G

→ these vertices and edges can never be in an $(1+\varepsilon)$-approx. solution since they are more than $2 \cdot OPT$ away from any terminal

→ all shortest paths in G are now of length at most $4 \cdot OPT$.

(4) Uncontract r and set $T = T_0 \cup SPT$
$\rightarrow$ T is a spanning tree of G

(5) find a spanning tree T^* in $G^* - E(T)$.
$\rightarrow$ T^* is a spanning tree of G^*

(6) Let $X = E(G) - E(T) - E(T^*)$.
$\rightarrow$ (T, T^*, X) is a tree-cotree decomposition

(7) pick an arbitrary root $r' \in V(T_0)$ for T and define

$$CG := T_0 \cup \{ loop(T, e) : e \in X \}.$$

$\rightarrow$ CG is a cut graph for G and
$\ell(CG) \leq 2 \cdot OPT + g \cdot 8\,OPT = (8g+2) \cdot OPT$

(8) Cut the graph open along CG to obtain a planar graph G_P with a distinguished face f_{outer} that contains all terminals.
$\rightarrow$ $\ell(f_{outer}) = 2\,\ell(CG) \leq (16g+4) \cdot OPT$

(9) Proceed as in planar spanner construction!
Find strips, columns, super-columns, mortar graph, bricks, portals, and add all possible Steiner trees among portals. $\rightarrow$ see Lecture 16
Crucial point: The bricks are planar and hence our planar structure theorem suffices!

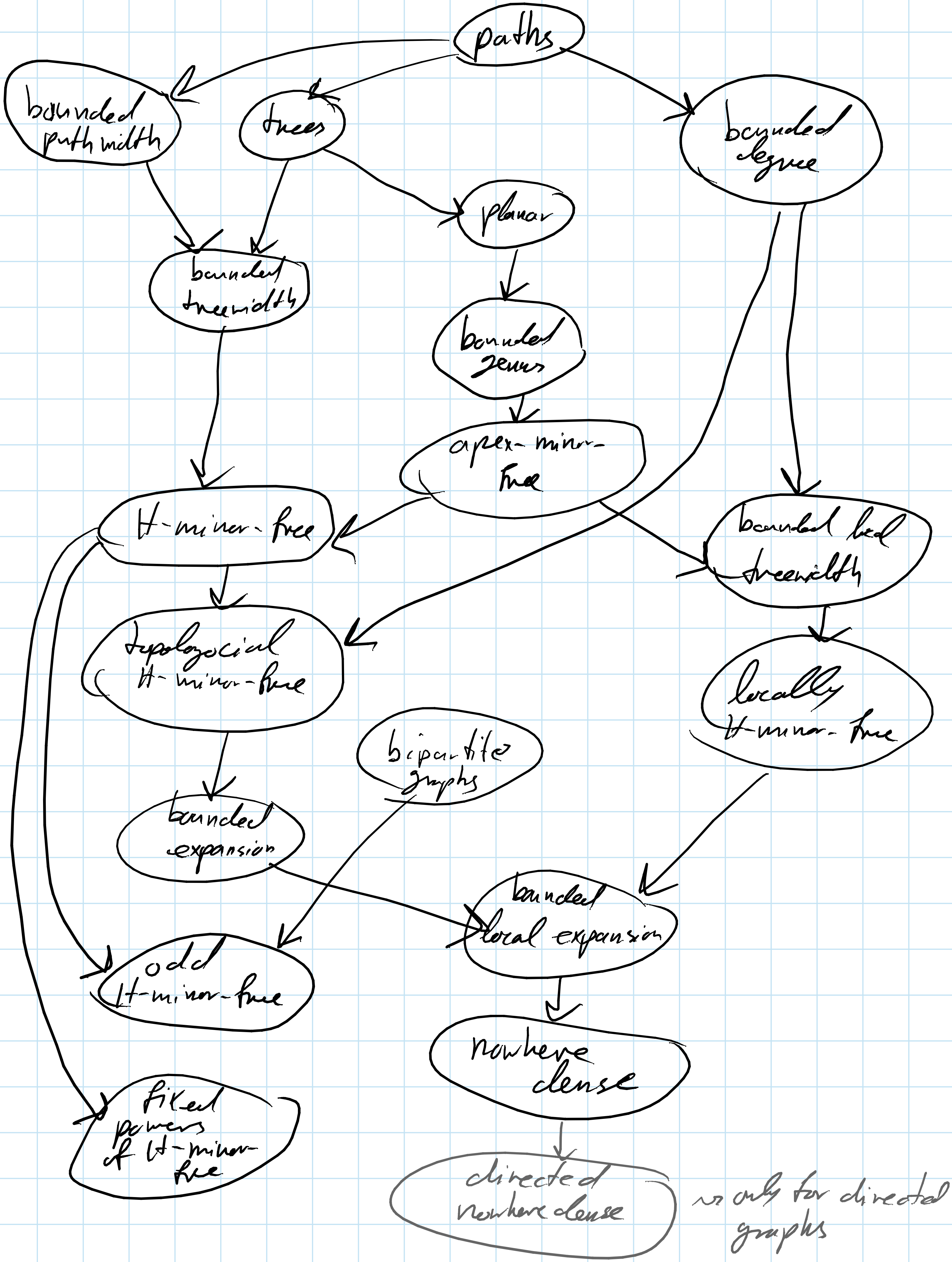
PTAS for Steiner tree in bounded-genus graphs:

① find spanner as above

② apply contraction-decomposition to spanner → Lecture 23

③ solve on bounded treewidth

④ Uncontract the small part contracted in ② to obtain solution for original graph.

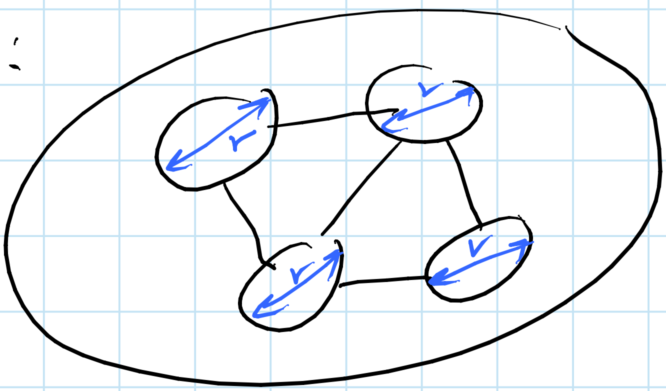


paths
bounded pathwidth
trees
bounded degree
planar
bounded treewidth
bounded genus
apex-minor-free
H-minor-free
bounded local treewidth
topological H-minor-free
locally H-minor-free
bipartite graphs
bounded expansion
bounded local expansion
odd H-minor-free
nowhere dense
fixed powers of H-minor-free
directed nowhere dense
~> only for directed graphs

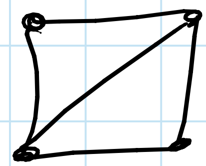
Shallow minors:

We say H is a depth-r minor of G, $H \preceq_r G$ if G contains a model of H in which every branch-set has diameter at most r:

G:



H:



Hence, depth-0 minors are subgraphs
depth-n minors are minors.

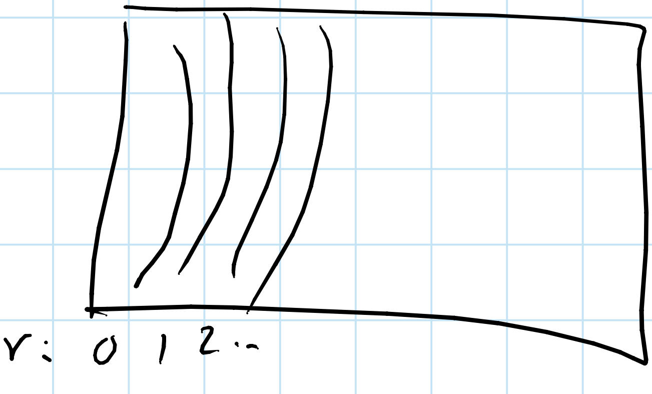
For a class of graphs $\mathcal{C}$ define

$\mathcal{C} \nabla r = \{$ the set of all depth-r minor of graphs of $\mathcal{C}\}$.

We say $\mathcal{C}$ is somewhere dense if there exists an r such that $\mathcal{C} \nabla r$ is equal to the set of <u>all</u> graphs (i.e. contains arbitrarily large cliques).

$\mathcal{C}$ is nowhere dense if it is not somewhere dense, i.e. for every r, there exists a k such that $\mathcal{C} \nabla r$ does not contain K_k.

Pool of all graphs:



→ If $\mathcal{C}$ is nowhere dense, this pool never fills up!

We say $\mathcal{C}$ has bounded expansion if there is a function f, s.t. every graph in $\mathcal{C} \nabla r$ has average degree at most $f(r)$.

First-order logic – is FPT on bounded-expansion
– is not FPT on somewhere dense
– on nowhere dense: OPEN!

However, subgraph isomorphism, dominating set, etc. are FPT on nowhere dense.

For a nowhere dense class $\mathcal{C}$ and given k, there exists a t ($\gg k$) s.t. every graph in $\mathcal{C}$ can be colored by t colors, s.t. any k colors induce a graph of bounded treewidth.

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