

Contraction decomposition:

[Demaine, Hajiaghayi, Kawarabayashi - STOC 2011]

for any specified  $k \geq 2$ :

any  $H$ -minor-free graph  $G = (V, E)$

has an edge partition  $E = E_1 \cup E_2 \cup \dots \cup E_K$   
such that  $\text{tw}(G/E_i) \leq c_H \cdot k$  for all  $i$

↳ contraction

- saw planar case in L15 [Klein - FOCS 2005]
- focus here on bounded-genus case  
[orig. by Demaine, Hajiaghayi, Mohar - SODA 2007]

Motivation: like generalized Baker's approach  
(deletion decomposition - L9)  
but good for contraction-closed problems

PTAS on  $H$ -minor-free graphs

for any minimization problem that:

① closed under contractions

② polynomial on graphs of bounded treewidth

③ spanner:  $G \rightarrow G'$ ,  $w(G') = O(\text{OPT}(G'))$   
solution  $S' \rightarrow S$ ,  $\text{cost}(S) \leq (1+\delta) \text{cost}(S')$

④ expand solution  $\hat{S}$  to  $G/X \rightarrow S$  to  $G$ ,  
 $\text{cost}(S) \leq \text{cost}(\hat{S}) + O(w(X))$

↳ input

Proof: ③  $\Rightarrow G \rightarrow G'$  ↗ to be specified  
 Contraction Decomp  $(G', k) \Rightarrow E' = E_1 \cup \dots \cup E_K$   
 Shifting  $\Rightarrow w(E_i) \leq \frac{1}{k} w(E')$  for some  $i$   
 $= O\left(\frac{1}{k} \text{OPT}(G')\right)$

- ②  $\Rightarrow \hat{S}$  optimal on  $G/E_i$
- ①  $\Rightarrow \text{cost}(\hat{S}) \leq \text{OPT}(G)$
- ④  $\Rightarrow \hat{S} \rightarrow S', \text{cost}(S') \leq \text{cost}(\hat{S}) + O(w(E_i))$   
 $\leq \text{OPT}(G) + O\left(\frac{1}{k} \text{OPT}(G')\right)$
- ③  $\Rightarrow S' \rightarrow S, \text{cost}(S) \leq (1+\varepsilon) \text{OPT}(G)$   
 for  $k = O\left(\frac{1}{\varepsilon}\right), S = O(\sqrt{\varepsilon})$   $\square$

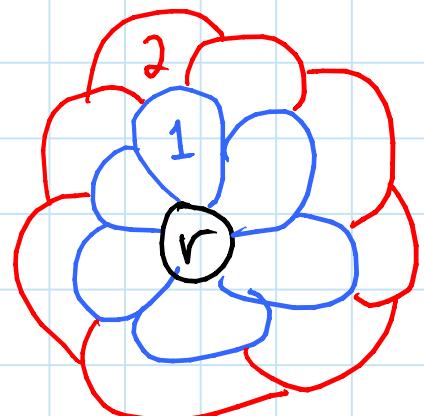
FPT algorithm for  $k$ -cut in  $H$ -minor-free graphs  
remove fewest edges to make  $\geq k$  connected comp's

- average degree  $\leq c_h = O(|V(H)| \sqrt{\log |V(H)|})$
- $\Rightarrow \text{OPT} \leq c_h \cdot k$  \* \* \* \*
- contraction decomp. with  $c_h \cdot k + 1$  parts
- $\Rightarrow$  some  $E_i$  misses OPT
- solve all  $G/E_j$ 's (bounded treewidth)
- test solution in  $G$
- generalized to arbitrary graphs  
[Kawarabayashi & Thorup 2011]

# Radial coloring: bounded-genus solution

- root vertex  $r$
- BFS in radial graph  
*(vertex-face incidence)*
- vertex layer  $j = \text{step } 2^j$
- $E_i = \bigcup_{j \equiv i \pmod{k}} \text{layer } j$

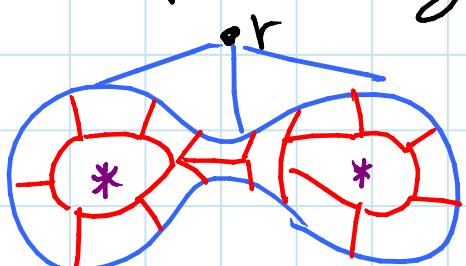
**SIMPLE!**



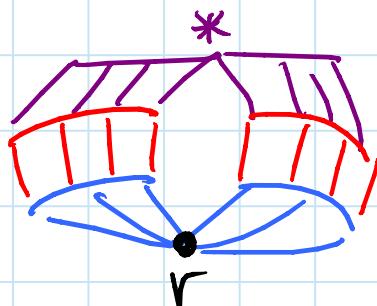
Fact:  $\leq 2$  distinct colors on edges of face

## Analysis:

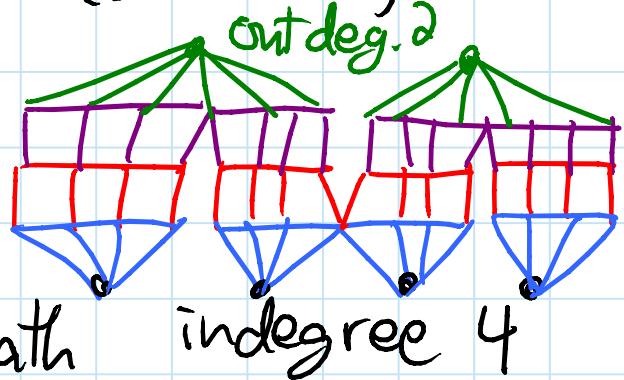
- $G/E_i$  contracts each connected component of layer  $j \equiv i$  into articulation point
- $G'$  = split each art. point into two  
(one connected to layers  $< j$ , other  $> j$ )
- blob = connected component of  $G'$
- blob DAG: vertex = blob  
edge = articulation pt. ( $\text{low} \rightarrow \text{high}$ )
- outdegree  $> 1$  = split  
(happens even in planar graphs)



- indegree  $> 1$  = rejoin
  - $\Rightarrow$  handle
  - $\Rightarrow$  only  $g$  times
  - $\Rightarrow$  indegree  $\leq 1 + g$

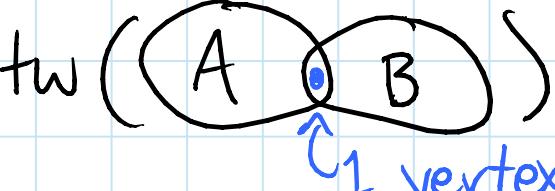


- partition blob into  $\leq 1+g$  chunks according to which articulation pt. is closest radially (BFS seed)  $\Rightarrow \leq 2k$   
 $\Rightarrow$  each chunk has radial diameter  $\leq 4k$  (to art. pt. & back)
- chunks connected in blob, at worst in a path  
 $\Rightarrow$  blob has radial diameter  $\leq (4k+1)(g+1)$   
 $\Rightarrow$  blob has treewidth  $O(kg)$  [Eppstein 2000]  
 (linear local treewidth)



- blob DAG is tree + g extra connections
- tree of blobs has treewidth  $O(kg)$ :  

$$\text{tw}((\text{A} \cap \text{B})) = \max\{\text{tw}(\text{A}), \text{tw}(\text{B})\}$$



  - g extra connections increases treewidth by  $\leq g$ : add those art. pts. to all bags  
 $\Rightarrow$  treewidth =  $O(kg)$

**OPEN:** better?  $O(k+g)$ ?

Generalization: root set  $R >$  one vertex

-  $c$  connected components of  $R$

$\Rightarrow \leq c+g$  indegrees beyond 1

$\Rightarrow \text{treewidth} = O(k(c+g))$

- root blobs (containing conn. comp. of  $R$ )

also different: no seed/diameter bound

- need connected components of  $R$   
have bounded local treewidth

$\Rightarrow \text{treewidth} = O(k(c+g) + \text{ltw}(R, k))$

-  $R$  tight if bounded ltw &  $c = O(1)$

- e.g.  $R = O(1)$  vertices

- e.g.  $R = O(1)$  radially shortest paths  $\leftarrow$  HARD

Generalization: special face with  
uncontractible edges

- only messes up one layer  $j \equiv i \pmod{k}$

- still contract all even or odd layers  
of  $E_i$

$\Rightarrow$  double  $k$  to get treewidth bound

Generalization:  $q$  special faces

- multiply  $k$  by  $q+1$

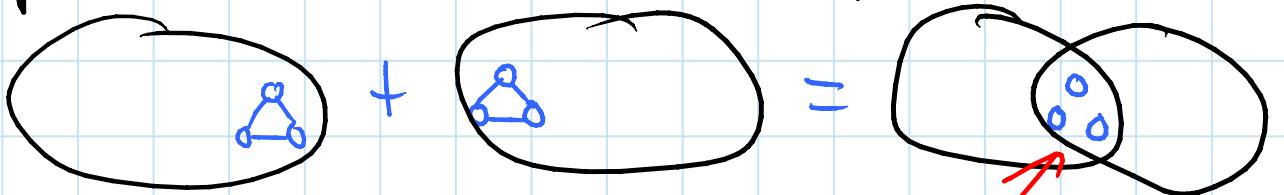
- one class of layers still contracted

$\Rightarrow \text{treewidth} = O(k(q+1)(c+g) + \text{ltw}(R, k))$

oblivious  
which

## $h$ -almost-embeddable graphs:

- $h$  apices: add to all bags  
     $\Rightarrow$  treewidth increases by  $\leq h$
- $h$  vortices: pasting each into a face  
    increases treewidth by  $O(h^2)$
- clique-sums are the hard part



$\Rightarrow$  desired contractions don't happen :

- good news: clique-sum either  
    vortex  $\leftrightarrow$  apices  $\Rightarrow$  no contraction necc.  
    or  $\leq$  triangle of bounded-genus  $\leftrightarrow$  apices

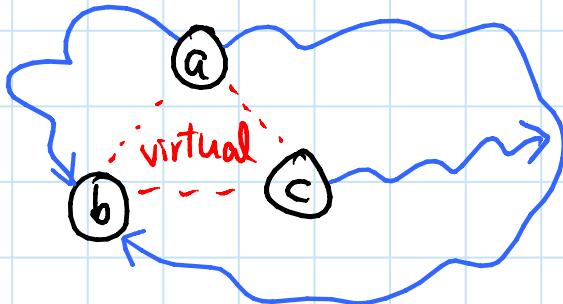
Idea: Simulate  $\leq 3$  missing edges by  
edge-disjoint paths (in children parts)

- can make parts 3-edge-connected
- but how to force entire path to be  
    colored uniformly?

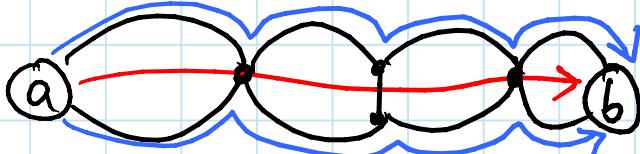
Idea: make paths root of radial coloring

- $\Rightarrow$  all the same color.  $\otimes$
- need to make them radially shortest  
 $\Rightarrow$  tight

- actually find  $\bar{\tau}$  ("te") structure:



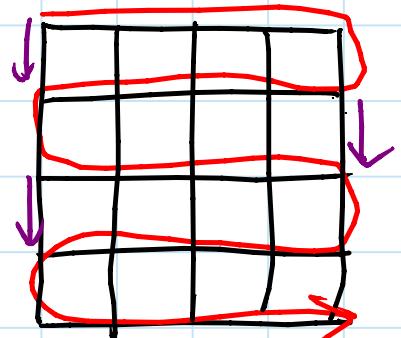
- shortest radial path  $a \rightarrow b$   
carved into two radially shortest paths



- shortest radial path  $c \rightarrow$  other paths

Radially shortest paths are tight: intuition

- large ltw  $\Rightarrow$  large grid
- all within  $r = O(1)$  of path
- but then must be  
shortcuts  $\Rightarrow$  not shortest



- not so easy: grid minor
- actual proof uses "dives" & "rainbows"