Graphs on Surfaces

A surface is a connected, compact, Hausdorff topological space $S$ which is locally homeomorphic to an open disc in the plane, i.e., each point of $S$ has an open neighborhood homeomorphic to the open unit disc in $\mathbb{R}^2$.

Let $F$ be a finite collection of pairwise disjoint convex polygons in the plane (including their interiors) with all sides of length 1.

- Suppose all these polygons together have sides $\overline{0_1, \ldots, 0_m}$ where $m$ is even.
- Choose an arbitrary orientation for each $0_i$.
- Choose an arbitrary partition of $\{0_1, \ldots, 0_m\}$ into pairs.
- Identify each pair so that the orientations agree.

$\Rightarrow$ we obtain a surface $S$. 
The sides $v_1, \ldots, v_m$ and their endpoints determine a connected multigraph $G$ embedded in $S$.

We say $G$ is 2-cell embedded in $S$.

The polygons in $F$ are the faces of $G$.

If all faces are triangles, we say $G$ triangulates $S$ and $S$ is called a triangulated surface.

**Theorem:** Every surface is homeomorphic to a triangulated surface.

**Proof Idea:** Use compactness! (See book)

- Consider two disjoint triangles $T_1$ and $T_2$ (i.e., all the sides have some length) in a face $F$ of a 2-cell embeddable multigraph $G$ in surface $S$.
- Obtain a new surface $S'$ by deleting the interiors of $T_1$ and $T_2$ and identifying $T_1$ and $T_2$ s.t. their clockwise orientations disagree.
- We say $S'$ is obtained from $S$ by adding a handle.
Define $S_h$ as a surface obtained by adding $h$ handles to the sphere $S$. $S_h$ is an orientable surface of genus $h$.

Define $N_h$ as a surface obtained by adding $h$ crosscaps to the sphere. $N_h$ is a non-orientable surface of genus $h$.

$S_1$: torus $S_2$: double torus

$N_1$: projective plane $N_2$: Klein bottle
Observe: If we already have a crosscap then adding a handle is same as adding twisted handle.

Proposition: Let $S$ be a surface obtained from the sphere by adding $h$ handles, $t$ twisted handles and $c$ crosscaps. If $t = c = 0$, then $S = S_h$. Otherwise, $S = N_{2h + 2t + c}$.

Theorem: Every surface is homeomorphic to precisely one of the surfaces $S_h$ ($h \geq 0$), or $N_k$ ($k \geq 1$).

Lemma: Let $S$ be a surface and $G$ a multigraph that is 2-cell embedded in $S$ with $n$ vertices, $m$ edges and $f$ faces. Then $S$ is homeomorphic to either $S_h$ or $N_k$ where $h$ and $k$ are defined by $n - m + f = 2 - 2h = 2 - k$.

Proof idea: triangulate $G$, cut away a triangle. Either get two triangles $\rightarrow$ handle or a hexagon $\rightarrow$ cross cap. (Must contain Möbius strip.)
Euler characteristic of a surface:

\[ X(S) = \begin{cases} 
2 - 2h & \text{if } S = S_h \\
2 - k & \text{if } S = N_k 
\end{cases} \]

Euler's formula: let \( G \) be a multigraph which is 2-cell embedded in a surface \( S \), has \( n \) vertices, \( m \) edges, and \( f \) faces. Then

\[ n - m + f = X(S). \]

The surface \( S_2 \)

The surface \( N_2 \)

Every surface has such a representation via its fundamental polygon.

An embedding of \( G \) in \( S \) is cellular if every face of \( G \) is homeomorphic to an open disc in \( \mathbb{R}^2 \).

Every 2-cell embedding is cellular. Conversely, every cellular embedding is homeomorphic to a 2-cell embedding.

→ see book for a proof
Embedding Schemes and Combinatorial Embeddings

Consider a cellular embedding of a connected multigraph into a surface $S$. For each vertex $v$, arbitrarily choose an orientation around it to be called clockwise.

To obtain a rotation system, i.e., a permutation $\Gamma$ on the set of darts of $G$ such that its orbits are the vertices of $G$.

For a dart $d$, define $\lambda(d) = 1$ if $d = uv$ and the clockwise orientation around $u$ and $v$ agree; otherwise let $\lambda(d) = -1$.

Note $\lambda(r(v, d)) = \lambda(d)$.

The function $\lambda: \text{Darts}(G) \to \{1, -1\}$ is called a signature. An embedding scheme $\mathcal{E} = (\Gamma, \lambda)$, where $\Gamma$ is a rotation system and $\lambda$ a signature.

If $\lambda(d) = 1$ for all darts $d$, then $S$ is orientable. Otherwise, obtain $\Gamma$-facial walks as follows: Whenever we traverse an edge with negative signature, we change from CW mode to CCW mode and vice versa. The walk ends when we encounter the same dart in the same mode.
Facial walks that are cyclic shifts of each other are considered the same. This can be weird as in the following example:

Consider projective plane:

\[ x(012) \Rightarrow x(012)_1, x(d)_{2-1} \]

Facial walks:

\[ 1) vau bwc v \Rightarrow b a h d \text{ in } (W) \]
\[ 2) v d u b w c v d u v \Rightarrow b a h d \text{ in } CCW \text{ mode} \]
\[ 3) u d v c w b u d v a u \Rightarrow b a h d \text{ in } CCW \text{ mode} \]

But walks 2 and 3 are the same! This embedding consists of only 2 faces:

\[ \text{Hom to argue that every edge appears exactly once in facial walks.} \]

In order to make this work with darts, we have to define traversing a dart in CCW mode as having traversed rev(d) – somewhat complicated!

Every embedding scheme for \( G \) defines a surface \( S \) on which \( G \) is 2-cell embedded. \( S \) is nonorientable iff \( G \) contains a cycle with odd \( 2t \) of edges with negative signature.
Otherwise, take a spanning tree $T$ of $G$, make signature positive on $T$, they will be positive everywhere.

Observation: • Every connected multigraph has a 2-cell embedding on some orientable surface.
• Every connected multigraph with at least one cycle has a 2-cell embedding on some nonorientable surface.

The genus $g(G)$ and the nonorientable genus $\tilde{g}(G)$ of a graph $G$ is the min. $h$ and the min. $k$, resp., s.t. $G$ has an embedding into $S_h$, resp. $N_k$.

Theorem: Every min. genus embedding of a connected graph is cellular.

Note: $\tilde{g}(G) \leq 2g(G)+1$ but $g(G)$ can be arbitrarily larger than $\tilde{g}(G)$

An embedding scheme $\Pi=(\Omega, d)$ defines combinatorial embedding of $G$. We say $G$ is $\Pi$-embedded.

Euler characteristic of $\Pi$, $\chi(\Pi)=n-m+1$

$\frac{1}{2} \chi(\Pi) = 1 - \frac{1}{2} \chi(\Pi)$

$\tilde{g}(\Pi) = 2 - \chi(\Pi)$

e$g(\Pi) = 2 - \chi(\Pi)$ Euler genus
Given a graph $G$ with embedding $\Gamma$, can define its dual $G^*$ with embedding $\Gamma^* = (\Gamma^*, \lambda^*)$:

- vertices of $G^*$ ~ $\Gamma$-facial walks
- edges of $G^*$ ~ edges of $G$

$\lambda^*(e) = 1$ iff the $\Gamma$-facial walks that contain edge traverse $e$ in opposite direction; i.e., $\lambda^*(e) = -1$.

$\Gamma^*$ orientable $\iff$ $\Gamma$ orientable

If $e$ is not a loop, then $(G^*/e)^* = G^*-e$.
If $e$ is contained in the facial walks, then $(G^*/e)^* = G^*-e$.

Cycles of embedded graphs

Cycle $C$ of $\Gamma$-embedded graph is $\Gamma$-one-sided if it has an odd number of edges with negative sign. Otherwise, it is $\Gamma$-two-sided.

Let $C$ be a $\Gamma$-two-sided cycle. Define the left graph $G_L (C, \Gamma)$ to be the part of $G$ that is connected to the left side of $C$. Similarly, define the right graph $G_R (C, \Gamma)$ (excluding $C$).
$C$ is \(\Pi\)-separating if $C$ is \(\Pi\)-two-sided and \(G_2(C)\) and \(G_r(C)\) have no edges in common.

$C$ is \(\Pi\)-contractible if one of \(G_\infty(C)\) or \(G_r(C)\) have \(\Pi\)-genus zero.

\[\begin{align*}
C & \sim \text{ contractible} \\
C_1 & \sim \text{ non-contractible separating} \\
C_2 & \sim \text{ non-contractible non-separating}
\end{align*}\]

If $C$ is \(\Pi\)-two-sided, let $\overline{G}$ be the graph obtained from $G$ by replacing $C$ with two copies of $C$ such that all edges on the left side of $C$ are incident with one copy of $C$ and all edges on the right side of $C$ are incident with the other copy. We say $\overline{G}$ is obtained from $G$ by cutting along $C$.

\[\begin{align*}
\overline{G} & \sim \text{ cutting along } \Pi\text{-two-sided cycle } C \\
C_1 & \sim \text{ cycle } C \text{ whose length is twice of } C
\end{align*}\]

Similarly, define cutting along \(\Pi\)-one-sided cycle $C$, $C$ is replaced by cycle $C$ whose length is twice of $C$.

\[\begin{align*}
\text{cutting along } \Pi\text{-one-sided cycle } C
\end{align*}\]
Lemma: Let $C$ be a $T$-nonseparating cycle of a $T$-emb.
graph $G$. Let $\bar{G}$ and $\bar{T}$ be obtained by cutting
along $C$. Then all $T$-facial walks are $\bar{T}$-facial
walks in $\bar{G}$ where edges of $C$ are replaced by their
copies of $C$ in $\bar{G}$.
If $C$ is $T$-twosided, then $eg(\bar{T}) = g(\bar{T}) - 2$
and the two copies of $C$ are the new $\bar{T}$-facial cycles.
If $C$ is $T$-onesided, then $eg(\bar{T}) = g(\bar{T}) - 1$ and
$C$ is a facial cycle in $\bar{G}$.

Enables induction on the genus until we
get planar — widely used technique!