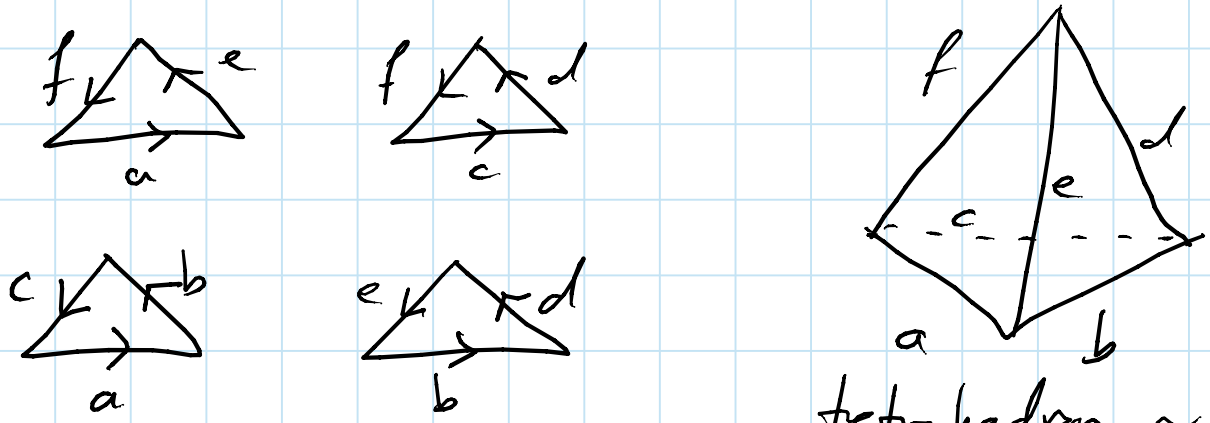
6.889	Lecture 22	Nov. 30, 2011

Graphs on Surfaces

A **surface** is a connected compact Hausdorff topological space S which is *locally* homeomorphic to an open disc in the plane, i.e. each point of S has an open neighborhood homeomorphic to the open unit disc in $\mathbb{R}^2$.

- let $\mathcal{F}$ be a finite collection of pairwise disjoint convex polygons in the plane (incl. their interiors) with all sides of length 1.
- suppose all these polygons together have sides $\sigma_1, \dots, \sigma_m$ where m is even.
- choose an arbitrary orientation for each σ_i
- choose an arbitrary partition of $\{\sigma_1, \dots, \sigma_m\}$ into pairs
- identify each pair s.t. the orientations agree

$\longrightarrow$ we obtain a surface S



tetrahedron $\sim K_4$
$\sim$ Sphere

The sides $\sigma_1, \ldots, \sigma_m$ and their endpoints determine a connected multigraph G embedded in S.

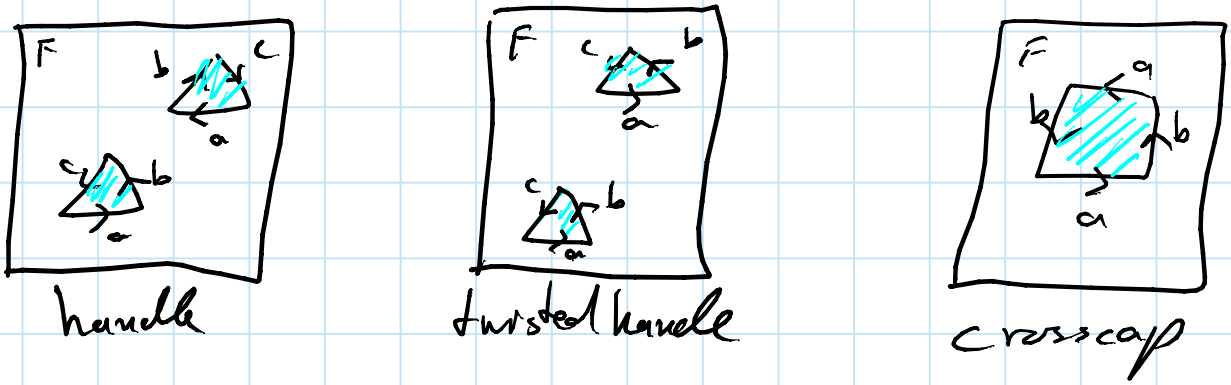
→ We say G is 2-cell embedded in S
- the polygons in $\mathcal{F}$ are the faces of G
- if all faces are triangles, we say G triangulates S and S is called a triangulated surface.

Theorem: Every surface is homeomorphic to a triangulated surface.

Proof Idea: Use compactness! (See book)

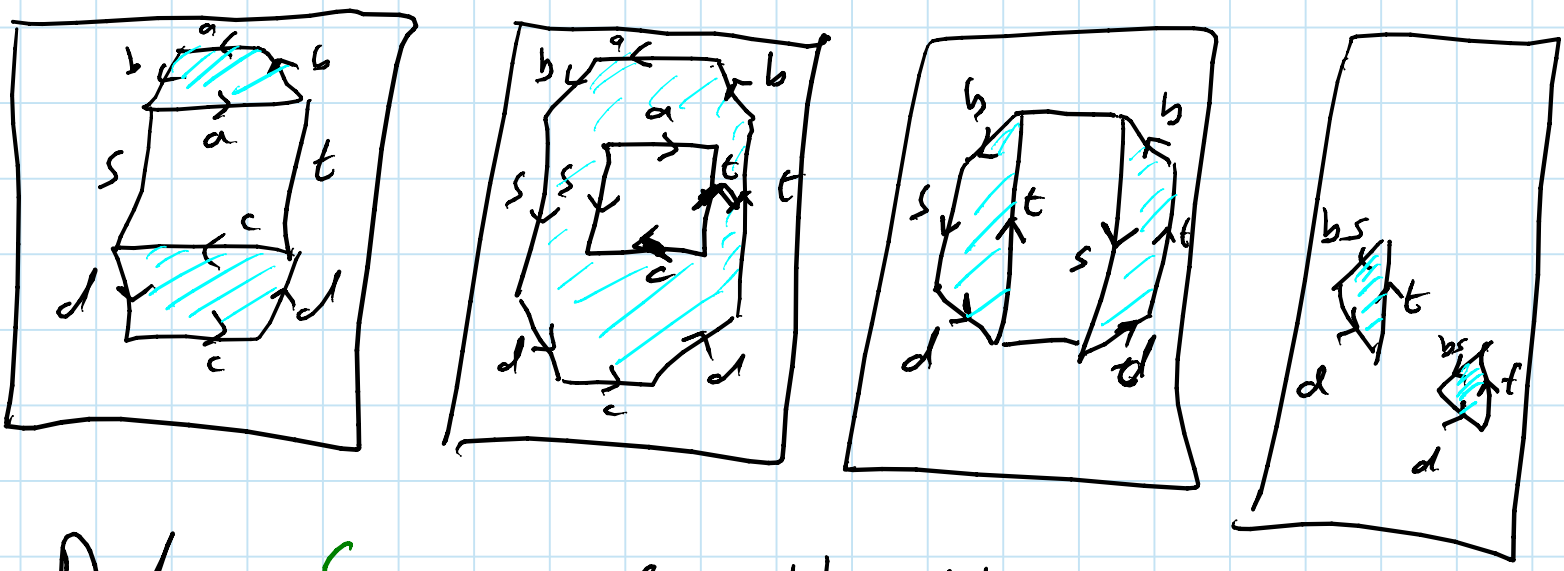
- Consider two disjoint triangles T_1 and T_2 (s.t. all the sides have same length) in a face F of a 2-cell embedd multigraph G in surface S.
- obtain new surface S' by deleting the interiors of T_1 and T_2 and identifying T_1 and T_2 s.t. their clockwise orientations disagree.

→ We say S' is obtained from S by adding a handle.



– independent of location of triangles, choice of F and even using different faces.

twisted handle ~ two crosscaps!



Define S_h as surface obtained by adding h handles to the sphere S_0. → orientable surface of genus h

Define N_h as surface obtained by adding h cross-caps to the sphere. → nonorientable surface of genus h

S_1 : torus $\qquad$ S_2 : double torus

N_1 : projective plane $\qquad$ N_2 : Klein bottle.

Observe: If we already have a crosscap then adding a handle is same as adding twisted handle.

Proposition: Let S be a surface obtained from the sphere by adding h handles, t twisted handles and c crosscaps. If $t=c=0$, then $S=S_h$. Otherwise, $S=N_{2h+2t+c}$.

$$\boxed{\text{Theorem: Every surface is homeomorphic to precisely one of the surfaces } S_h\ (h\geq 0), \text{ or } N_k\ (k\geq 1).}$$

Lemma: Let S be a surface and G a multigraph that is 2-cell embedded in S with n vertices, m edges and f faces. Then S is homeomorphic to either S_h or N_k where h and k are defined by

$$n-m+f=2-2h=2-k.$$

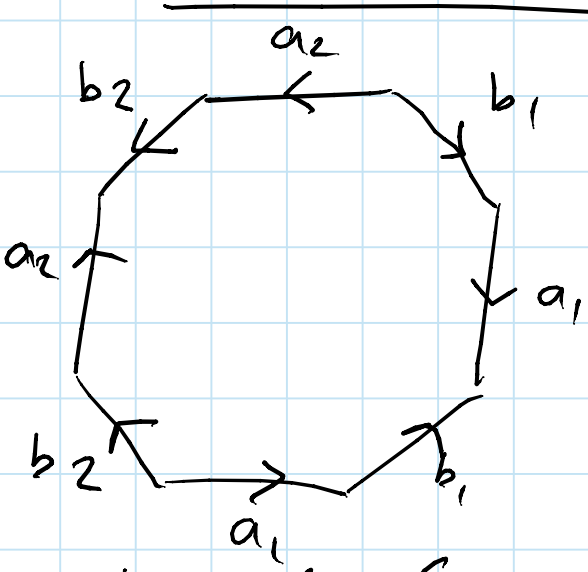
Proof idea: triangulate G. Cut along a triangle. Either get two triangles $\rightsquigarrow$ handle or a hexagon $\rightsquigarrow$ crosscap.
$\rightarrow$ (S contains Möbius strip).

Euler characteristic of a surface:

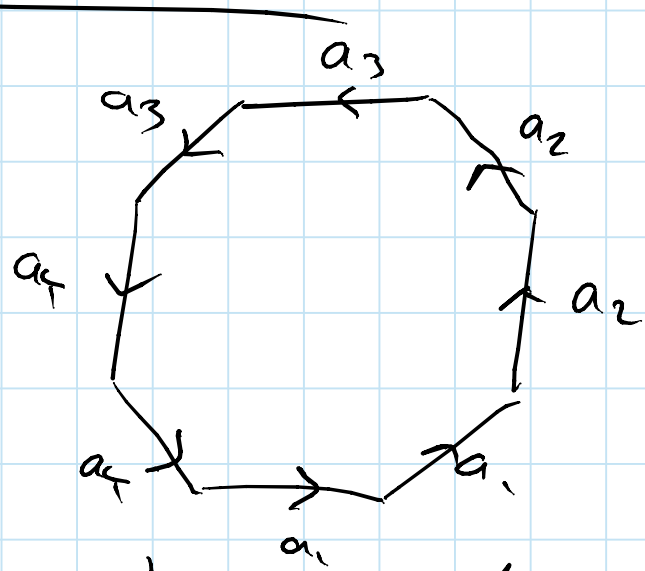
$$\chi(S) = \begin{cases} 2-2h & \text{if } S = S_h \\ 2-k & \text{if } S = N_k \end{cases}$$

Euler's formula: Let G be a multigraph which is 2-cell embedded in a surface S, has n vertices, m edges and f faces. Then

$$n - m + f = \chi(S).$$



The surface S_2



The surface N_2.

Every surface has such a representation via its fundamental polygon.

An embedding of G in S is cellular if every face of G is homeomorphic to an open disc in $\mathbb{R}^2$.

Every 2-cell embedding is cellular. Conversely, every cellular embedding is homeomorphic to a 2-cell embedding.

→ see book for a proof

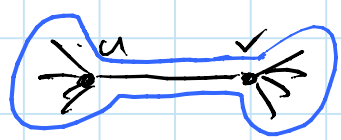
Embedding Schemes and Combinatorial Embeddings

Consider a cellular embedding of a connected multigraph into a surface S. For each vertex v, arbitrarily choose an orientation around it to be called <u>clockwise</u>.

→ obtain a rotation system, i.e. a permutation π on the set of darts of G such that its orbits are the vertices of G.

For a dart d, define $\lambda(d)=+1$ if $d=uv$ and the clockwise orientation around u and v agree; otherwise let $\lambda(d)=-1$.

Note $\lambda(\mathrm{rev}(d))=\lambda(d)$.



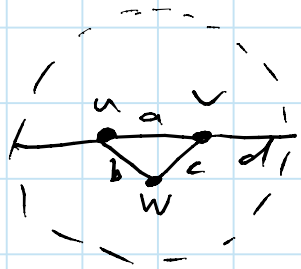
The function $\lambda : \mathrm{Darts}(G) \to \{+1,-1\}$ is called a signature. An embedding scheme $\Pi=(\pi,\lambda)$, where π is a rotation system and λ a signature.

If $\lambda(d)=+1$ for all darts d, then S is orientable. Otherwise, obtain Π-facial walks as follows: Whenever we traverse an edge with negative signature, we change from CW mode to CCW mode and vice versa. The walk ends when we encounter the same dart in the same mode.

Π-facial walks that are cyclic shifts of each other are considered the same.

This can be weird as in the following example:

Consider projective plane:



$\lambda(a) = \lambda(b) = \lambda(c) = 1$, $\lambda(d) = -1$

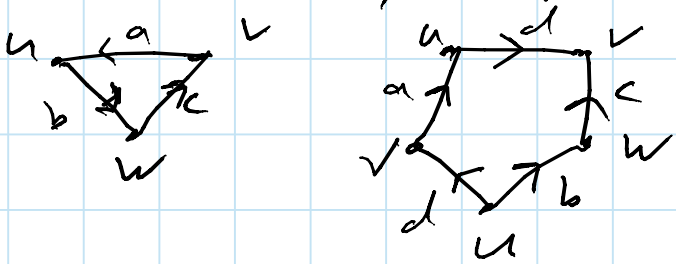
facial walks:

1) $vaubwcv$ → back to a in CW

2) $vd\underbrace{ubwcv}_{\text{CCW made}}dua v$ → back to d in CW made

3) $ud\underbrace{vcwbudv}_{\text{CCW made}}au$ → back to d in CW made

But walks 2 and 3 are the same! This embedding consists of only 2 faces:



Hard to argue that every edge appears exactly twice in facial walks.

In order to make this work with darts, we have to define traversing a dart in CCW mode as having traversed $rev(d)$.

⇝ somewhat complicated.

Every embedding scheme for G defines a surface S on which G is 2-cell embedded. S is nonorientable iff G contains a cycle with odd # of edges with negative signature.

Otherwise, take a spanning tree T of G, make signature positive on T $\rightarrow$ they will be positive everywhere.

Observation: • Every connected multigraph has a 2-cell embedding on some orientable surface.
• Every connected multigraph with at least one cycle has a 2-cell embedding on some nonorientable surface.

The genus $g(G)$ and the nonorientable genus $\tilde{g}(G)$ of a graph G is the min. h and the min. k, resp., s.t. G has an embedding into S_h, resp. N_k.

Theorem: Every min. genus embedding of a connected graph is cellular.

Note: $\tilde{g}(G) \leq 2g(G)+1$ but $g(G)$ can be arbitrarily larger than $\tilde{g}(G)$.

An embedding scheme $\Pi = (\sigma, \lambda)$ defines combinatorial embedding of G. We say G is Π-embedded.

Euler Characteristic of Π, $\chi(\Pi) = n - m + f$

$$g(\Pi) = 1 - \frac{1}{2}\chi(\Pi)$$

$$\tilde{g}(\Pi) = 2 - \chi(\Pi)$$

$$eg(\Pi) = 2 - \chi(\Pi) \quad \text{Euler genus}$$

connected multigraph

Given G with embedding $\Pi = (\pi, \lambda)$, can define its **dual** G^* with embedding $\Pi^* = (\pi^*, \lambda^*)$:

vertices of G^* $\sim$ Π-facial walks
edges of G^* $\sim$ edges of G

$\lambda^*(e) = 1$ iff the Π-facial walks that contain e traverse e in opposite direction; o.w. $\lambda^*(e) = -1$.

Π^* orientable $\iff$ Π orientable

If e is not a loop then $(G/e)^* = G^* - e$.
If e is contained in two facial walks, then $(G-e)^* = G^*/e$.

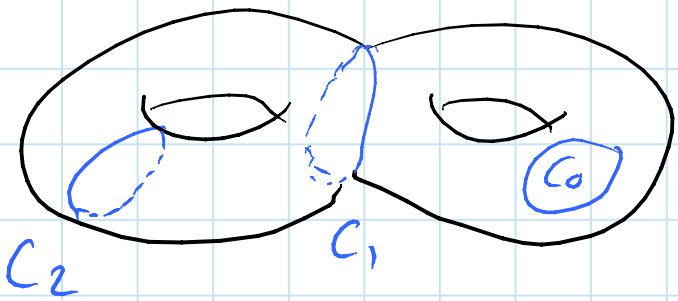
Cycles of embedded graphs

Cycle C of Π-embedded graph is **Π-one sided** if it has an odd # edges with negative sign. Otherwise, it is **Π-two sided**.

Let C be a Π-two sided cycle. Define the **left graph** $G_\ell(C, \Pi)$ to be the part of G that is connected to the left side of C. Similarly define the **right graph** $G_r(C, \Pi)$ (excluding C).

C is Π-separating if C is Π-two sided and $G_\ell(C)$ and $G_r(C)$ have no edges in common.

C is Π-contractible if one of $G_\ell(C) \cup C$ or $G_r(C) \cup C$ have Π-genus zero.

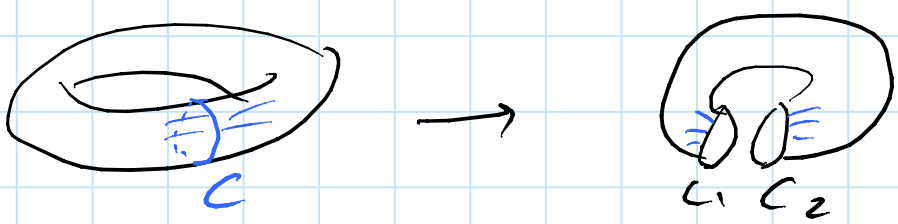


C_0 ~ contractible

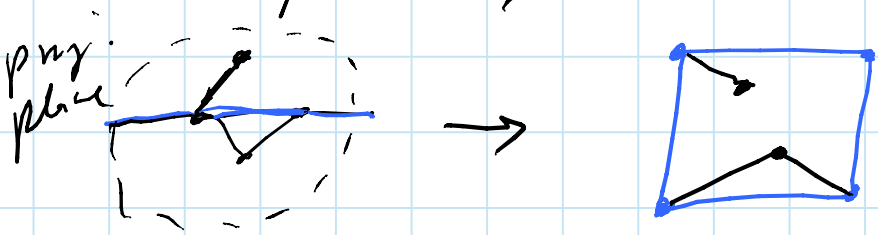
C_1 ~ non contractible separating

C_2 ~ non contractible non separating

If C is Π-two sided, let $\bar{G}$ be the graph obtained from G by replacing C with two copies of C such that all edges on the left side of C are incident with one copy of C and all edges on the right side of C are incident with the other copy.
We say $\bar{G}$ is obtained from G by cutting along C.



Similarly, define cutting along Π-one sided cycle C.
C is replaced by cycle $\bar{C}$ whose length is twice of C.



(see page 106 of book)

Lemma: Let C be a Π-nonseparating cycle of a Π-emb. graph G. Let $\bar{G}$ and $\bar{\Pi}$ be obtained by cutting along C. Then all Π-facial walks are $\bar{\Pi}$-facial walks in $\bar{G}$ where edges of C are replaced by their copies of C in $\bar{G}$.

If C is Π-twosided, then $eg(\bar{\Pi}) = eg(\Pi) - 2$ and the two copies of C are the new $\bar{\Pi}$-facial cycles.

If C is Π-onesided, then $eg(\bar{\Pi}) = eg(\Pi) - 1$ and C is a facial cycle in $\bar{G}$.

$\hookrightarrow$ enables induction on the genus until we get planar $\longrightarrow$ widely used technique!