

in the next 3 lectures we will discuss the min-cut and max-flow problems in planar graphs.

we will see beautiful connections between these problems and the shortest paths problem.

Let's start with min-cut. the setting:

$G = (V, E)$ undirected planar embedded graph
 $c: E \rightarrow \mathbb{R}^+$ weight function on edges
 $s, t \in V$

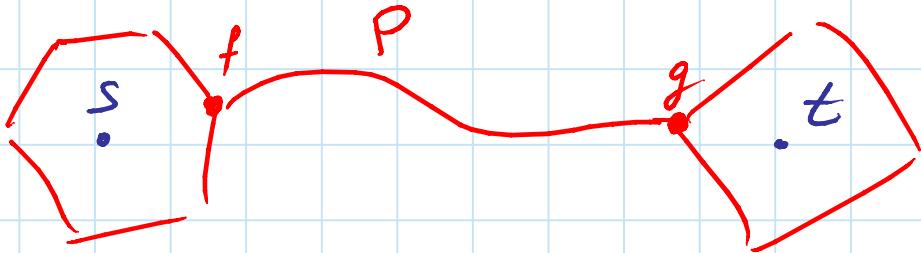
consider a cut $(S, V \setminus S)$ s.t. $s \in S, t \notin S$.
recall $\delta(S)$ is set of edges crossing the cut.
we want to find a cut that minimizes

$$c(\delta(S)) = \sum_{e \in \delta(S)} c(e)$$

[Reif 1983]

Reif's algorithm: assume wlog that G is connected. A min-cut is simple (i.e. both sides of the cut are connected), so in the dual G^* it corresponds to a minimum

length simple cycle that separates s and t .
 (here we consider $c(e)$ as the length of e in the dual)

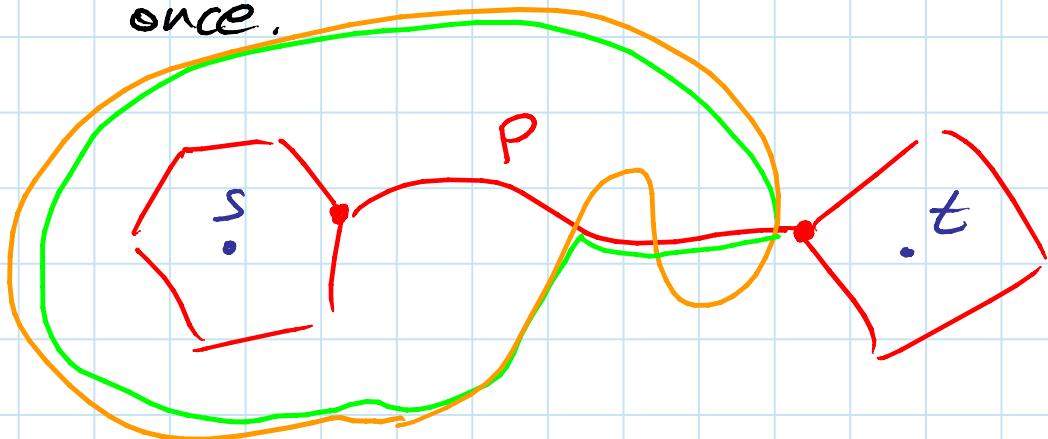


let f be some face incident to s ,
 let g be some face incident to t

let P be a shortest f -to- g path in G^* .

Observations:

- ① any cycle separating s and t crosses P
- ② there exists a min-length cycle that separates s and t and crosses P exactly once.

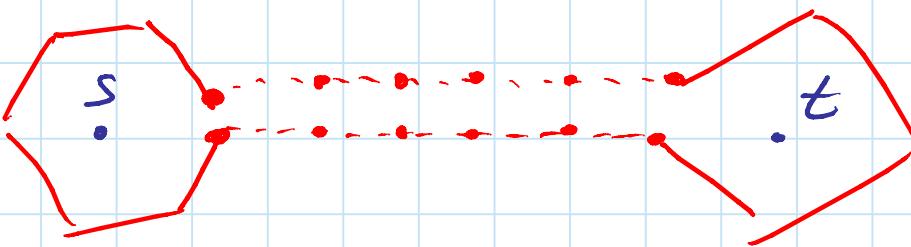


to find such a min-length cycle, cut G^* open along P . Now there are two copies of every node of P .

use MSSP to compute the SP between every pair of copies of each node of P .

the shortest such path defines the min-length cycle separating s and t in G^* .

The corresponding primal cut is a min s-t cut.



Running time: find P in $O(n)$

MSSP in $O(n \log n)$

number of pairs $O(n)$

so overall $O(n \log n)$.

notes: - MSSP not necessary - can do divide and conquer and still get $O(n \log n)$ time

- currently best known is $O(n \log \log n)$ using this overall idea, but r-decomp. and FR-Dijkstra [Italiano, Nussbaum, Sankowski, Wulf-Nielsen '11]

Given the techniques we already saw in class, Reif's algorithm is elegant, but perhaps not very surprising.

We now move on to flow

Flow basics:

$G = (V, A)$ directed connected planar graph

a flow assignment is an anti-symmetric function on the dart set D of G , $f: D \rightarrow \mathbb{R}$

$$f(d) = -f(\text{rev}(d))$$

capacity $C: D \rightarrow \mathbb{R}$

we are used to non-neg. capacities on arcs $c(a)$.
extend to darts by $c((a, +1)) = c(a)$
 $c((a, -1)) = 0$

a flow assignment f respects capacities if

$$f(d) \leq c(d) \quad \forall d \in D$$

so capacity is an upper bound on flow

what is the meaning of negative capacity?

$$f(d) \leq c(d) \Rightarrow f(\text{rev}(d)) \geq -c(d)$$

so neg. capacity on d is a lower bound on the flow on $\text{rev}(d)$. This will be useful when dealing with residual graphs.

f satisfies conservation at $v \in V$ if

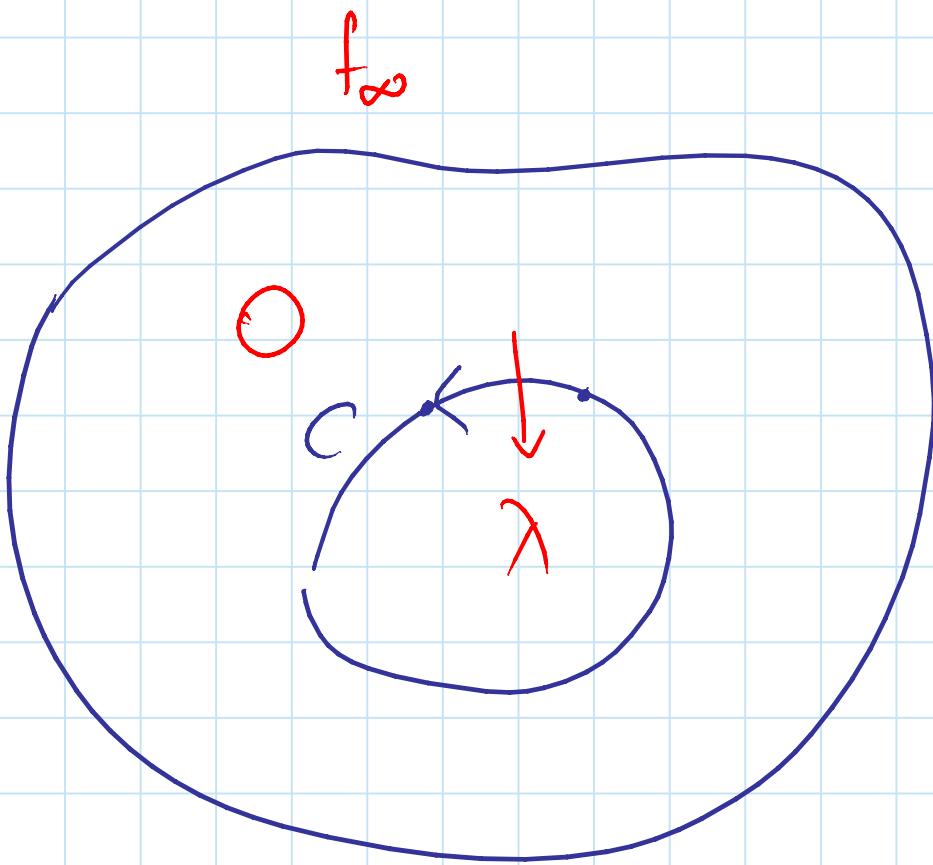
$$\sum_{d: \text{tail}(d)=v} f(d) = 0$$

a flow assignment that satisfies conservation at all nodes is called a circulation.

any circulation can be decomposed into a set of simple flow cycles.

Face prices and circulations:

Suppose Θ is a circulation that consists of a single simple CCW cycle of flow. That is, a CCW cycle C , each of whose darts carries the same amount λ of flow.



Define face prices φ by:

$$\varphi(g) = \begin{cases} \lambda & \text{if } g \text{ is enclosed by } C \\ 0 & \text{otherwise} \end{cases}$$

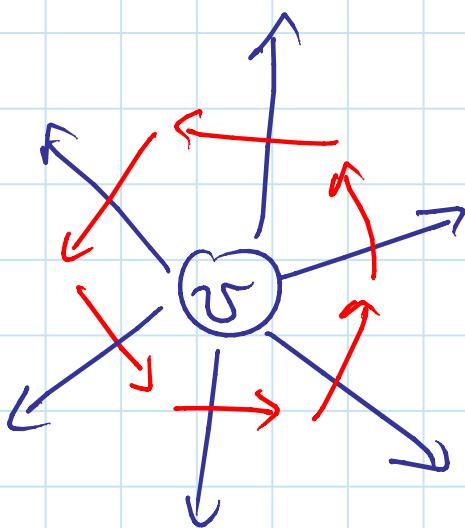
Then the relation between the circulation Θ and the face prices φ is given by

$$\begin{aligned}
 (\#) \quad \Theta(d) &= \varphi(\text{face to the left of } d) \\
 &\quad - \varphi(\text{face to the right of } d) \\
 &= \varphi(\text{head}(d^*)) - \varphi(\text{tail}(d^*))
 \end{aligned}$$

This relation extends to any circulation θ since θ can be decomposed into a set of simple cycles of flow.

So we see that circulation \Rightarrow face prices
 In fact, the opposite is also true.

Let φ be a price function on faces. We want to show that θ defined in (*) is a circulation.
 Consider a node v :



$$\sum_{d: \text{tail}(d)=v} \theta(d) = \sum_{d: \text{tail}(d)=v} \varphi(\text{head}(d^*)) - \varphi(\text{tail}(d^*))$$

$= 0$ telescopes out.

So face prices \Leftrightarrow circulations

Notes:

- According to our convention $\varphi(f_\infty) = 0$.
- residual capacity after pushing Θ is
 $c(d) - \Theta(d) = c(d^*) + \varphi(\text{tail}(d^*)) - \varphi(\text{head}(d^*))$.
which is the reduced length of d^* w.r.t. φ !
- What if face prices happen to be a feasible price function for G^* w.r.t. C ?

recall φ is feasible if for every dart,

$$\begin{aligned} & c(d) + \varphi(\text{tail}(d)) - \varphi(\text{head}(d)) \geq 0 \\ \Leftrightarrow & \varphi(\text{head}(d)) - \varphi(\text{tail}(d)) \leq c \\ \Leftrightarrow & \Theta(d) \leq c \end{aligned}$$

in fact, we see that a circulation respects capacities iff the corresponding face prices are feasible for G^* w.r.t. C .

Corollary: \exists circulation that respects capacities
iff there are no negative length cycles in G^*

Pf: (\leftarrow) no neg. cycles \Rightarrow SP well defined. SP distances form a feasible price function \Rightarrow defines a circulation that respects capacities

\Rightarrow \exists circulation that respects capacities \Rightarrow
corresponding price function is feasible
 \Rightarrow reduced length of any cycle is non-neg.
But for any cycle, reduced length equals
original length. So no neg.-length cycles

We see that we can solve the circulation problem.

Is there a circulation that respects capacities?

by one shortest-paths computation (w. neg. lengths)
in the dual.

st - flows :

Given $s, t \in V$ we want to find a flow assignment f s.t :

- (1) respects capacities
- (2) satisfies conservation at all nodes except s, t
- (3) maximizes the amount of flow leaving s : $\sum_{d: \text{tail}(d)=s} f(d)$
(called the value of f , $|f|$)

Observe: if f, f' are two st-flows with same value then $f-f'$ is a circulation.

the residual graph of G w.r.t. f is the same graph, equipped with residual capacities

$$c_f(d) = c(d) - f(d)$$

Suppose we knew value of max-flow is λ , how to find flow?

let f be any st -flow of value λ .

f does not have to respect capacities. E.g., route λ units of flow along any s -to- t path.

consider residual graph G_f . it might have negative capacities.

find a capacity respecting circulation θ in G_f using SP in the dual.

$$\theta(d) \leq C_r(d) = C(d) - f(d)$$

$$so \quad f + \theta \leq C$$

so $f + \theta$ is assignment that respects capacities, obeys conservation everywhere except s, t and has value λ .

But we do not know value of max-flow,

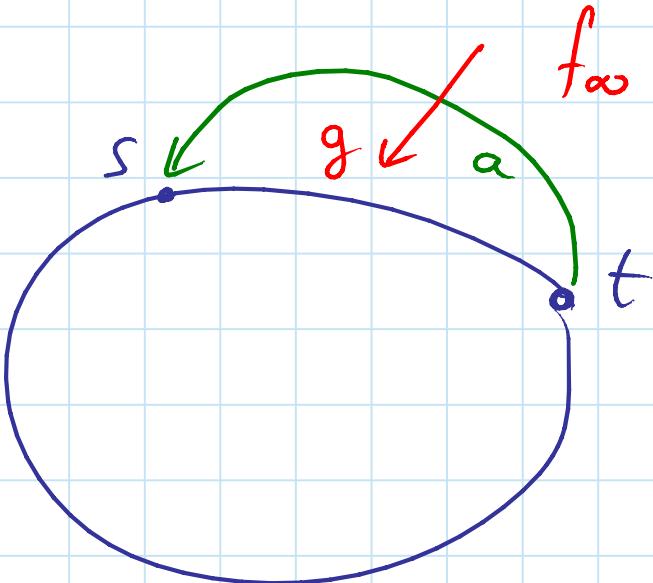
can binary search for it.

time:

$$O(\underbrace{n \lg n}_{\text{SP w/ neg. lengths}} \cdot \log C)$$

largest capacity

max-flow when s, t on same face (st-planar graph)



- 1) Add ∞ -capacity arc a from s to t (preserves planarity, introduces new face g)
- 2) Compute SP tree rooted at $\text{tail}(a^*)$ in dual f_∞
- 3) Use distance labels $\text{dist}(f_\infty, \cdot)$ as face prices to define corresponding circulation Θ .
- 4) return st-flow Θ' obtained from Θ by setting $\Theta'(d) = 0$ for both darts of a .

Correctness: we know that Θ is a circulation that respects capacities, so Θ' is an st-flow that respects capacities.

it remains to show maximality.

Since $c(a) = \infty$, it suffices to show that θ maximizes the flow on a among all capacity-respecting circulations.

Since every capacity respecting circulation corresponds to a feasible price function on the faces, it suffices to show that SP distances from $\text{tail}(a^*)$ maximize price of artificial face g among all feasible price functions normalized to $\Psi(f_\infty) = 0$.

Lemma: If $\text{dist}(f_\infty, v) \geq \Psi(v)$ for any feasible price function Ψ s.t. $\Psi(f_\infty) = 0$

Proof: By induction on the depth of v in the SP tree.
base case $v=r$ is trivial.

let u be the parent of v in SP tree.

By inductive hypothesis, $\text{dist}(f_\infty, u) \geq \Psi(u)$

Since Ψ is feasible, $c(uv) + \Psi(u) - \Psi(v) \geq 0$
so, $\Psi(v) \leq c(uv) + \Psi(u)$. But

$\text{dist}(f_\infty, v) = c(uv) + \text{dist}(f_\infty, u) \geq c(uv) + \Psi(u)$ □

Refs:

- Venkatesan's PhD thesis (1983)
- Hassin (1981)
- Miller, Naor (1995)
- khuller, Naor, Klein (1993)