Recall Mortar Graph–Brick Decomposition

Spurious construction selects a constant number of portals on each brick and adds Steiner trees for every subset of portals.

Goal: Show that any forest inside a brick can be transformed into another forest that spans the same vertices on the boundary but has few joining vertices and approx. the same weight.

Def.: Let \( H \subseteq G \) and \( P \) be a path in \( G \). A joining vertex of \( H \) with \( P \) is a vertex of \( P \) that is an endpoint of an edge of \( H - E(P) \).
Properties of a brick:

- terminals only exist on Nor S
- N is a shortest path
- every proper subpath of S is (1+\varepsilon)-short
- \( l(W) \) and \( l(E) \) are negligible

(total weight of all super-columns < \( \varepsilon \cdot 0.05 \))

\[ S = S_1 \circ S_2 \circ \ldots \circ S_k \] where, for each vertex \( x \in S_\cdot \cdot \cdot \):

\[ l(S; [\cdot, x]) \leq \varepsilon \cdot \text{dist}(x, N) \]

\[ K = \mathcal{O}(\varepsilon^{-3}) \]

Main strategy (very rough):

Case 1:

Same for \( S \)

Case 2:

Keep \( r \), simplify \( N \) and \( S \) separately

Case 3:

Keep \( r \) and \( t \)
In the following 3 lemmas:

- B a planar embedded graph
- K a subgraph of B
- P an (1+\varepsilon)-short path on the boundary \( \partial B \).

**Lemma 1:** There is a procedure \( \text{Span}_o (P; K) \) that returns a subpath of \( P \) spanning \( V(K) \cap V(P) \) whose total length is at most \( (1+\varepsilon) \cdot l(K) \).

**Proof:** \( l(K) \geq l(Q) \geq \text{dist}_G(u,v) \geq \frac{\text{dist}_P(u,v)}{1+\varepsilon} \).

**Lemma 2:** There is a procedure \( \text{Span}_i (P; K; r) \) for \( r \in V(K) \) that returns a subgraph of \( PVK \) such that:

- its length is at most \( (1+4\varepsilon) \cdot l(K) \)
- it has at most \( 11 \varepsilon^{-1.45} \) joining vertices with \( P \).
- spans \( \{r\} \cup (PVK) \).

level \( k \) how to choose?
Claim: There is a tree $T$ that is
- rooted at $r$
- spans all vertices of $K_N$
- is binary
- $\ell(T) \leq (1 + \varepsilon) \cdot \ell(K)$.

Proof of Claim: Take a subtree of $K$ spanning $K_N$ and all leaves in $K_N$ and root at $r$. If any vertex has at least 3 children, apply the following transformation:

Replace subtree of $u$ by the $x$-$y$-subpath of $P$ plus the shortest $u$-to-$P$ path.

$$\ell(\text{replacement}) = \ell(P_{xy}) + \text{dist}_K(u, P) \leq (1 + \varepsilon) \ell(Q) + \ell(Q')$$

$$\leq (1 + \varepsilon) \cdot \ell(\text{subtree of } u).$$

Proof of Lemma 2:
Let a super-edge of $T$ be a maximal subpath of $T$ with internal vertices of degree 2. Endpoints of super-edges are called super-vertices. The level of a super-edge is the number of super-edges traversed from the root to its start.
So, we basically ignore vertices of degree 2 (except r).

Select a level \( k \) (to be determined) and transform as before:

Note: We cannot quite proceed to prove as in previous claim since there is no third path \( Q' \) to blame.

Problem: Super-edge \( e = uv \) is blamed twice!

Can we somehow make sure that the total weight of super-edges who are blamed twice is small?

Yes! Choose \( k \) appropriately!
A super-edge uv is called a zig-zag edge if when coming from the parent of v we change direction:

Let $L_i$ denote the set of all zig-zag edges of level $i$.

Note that we can always designate the bad super-edge $e$ of $A$ and $A'$ to be a zig-zag edge in $L_k$. Also, note that $A$ and $A'$ avoids all zig-zag edges at levels $> k + 1$, i.e., avoids $L_{k+2} \cup L_{k+3} \cup \ldots$.

Let $T'$ be the tree obtained after transformation:

$$l(T') \leq (1 + \varepsilon) \left[ \ell(T) + L_k - \left( L_{k+2} + L_{k+3} + \ldots \right) \right]$$

Claim: Let $k_0 = \lceil \log_\phi (\sqrt{5}/\varepsilon + 1) \rceil$ while $\phi$ is golden ratio. Then there exists $k \leq k_0$ such that $L_k \leq \varepsilon \ell(T) + L_{k+2} + \ldots + L_{k_0}$. 
Proof of Claim: Otherwise, for every \( k = 1, 2, ..., k_0 \), we have
\[
L_k > \varepsilon L(T) + L_{k+2} + L_{k+3} + ... + L_{k_0}
\]

By induction, one can show that \( L_1 > \varepsilon \cdot L(T) \cdot \text{Fib}(k_0) \) where \( \text{Fib}(k_0) \) is the \( k_0 \)th Fibonacci number, which is greater than \( \frac{1}{\varepsilon} \) by choice of \( k_0 \). Thus \( L_1 > \varepsilon L(T) \).

By choosing \( h \) according to claim, we have
\[
L(T) \leq (1 + \varepsilon)^2 \cdot L(T).
\]

Also, the number of super-vertices at level \( k \) is at most:
\[
2^k \leq 2 \cdot \log \phi \left( \frac{1}{2} + \frac{1}{\varepsilon} \right) < \frac{1}{1 - \varepsilon^{-2.5}}
\]
whenever \( \varepsilon < 1 \). Hence, we obtain desired bound on number of joining vertices.

---

Lemma 3: There is a procedure \( \text{Span}_2^2 \) (\( P, K, r, t \)) that returns a subgraph of \( PUK \) s.t.
- spans \( \{r, t\} \cup \{P, K\} \)
- its length is at most \( (1 + c_1 \varepsilon) L(K) \)
- has at most \( c_2 \varepsilon^{-2.5} \) joining vertices with \( P \).

---

\( \rightarrow \) see proof in book/paper
Theorem: Let $B$ be a plane graph with boundary $NUEUSUW$ satisfying the brick properties. Let $F$ be a set of edges of $B$. There is a forest $\tilde{F}$ of $B$ satisfying the following properties:

- (P1) If two vertices of the boundary are connected in $F$ then they are connected in $\tilde{F}$.

- (P2) The number of joining vertices of $\tilde{F}$ with $N$ and with $S$ is at most $2c_14 + \varepsilon^{-2.5}$.

- (P3) $\ell(\tilde{F}) \leq (1 + c_3) (\ell(F) + \ell(E) + \ell(W))$.

Proof: Define paths $\overline{P_1, P_2, \ldots, P_i}$ as follows:

- $\overline{P_{k+1}} := P_{k+1} := E$

- If $F\cup W$ has an $S$-to-$N$ path that originates in $S_i \cup \ldots$ and does not intersect $\overline{P_1, P_2, \ldots, P_{k+1}}$, let $P_i$ be the rightmost such path.

- Define $P_i = S_i \cup \text{start}(P_i)$ to $P_i$.
Main Idea:

- get rid of components that have leaves only on $N$ or only on $S$ by applying $Span_1$

More formally:

- Consider $F = W \cup \bigcup_{i=1}^{k} \overline{P_i}$

- Let $F'$ be a minimal forest thereof that contains $\bigcup_{i=1}^{k} \overline{P_i}$ and preserves connectivity away from boundary

Note that $\ell(F_i') \leq (1+\varepsilon)\ell(P_i)$

$\implies \ell(F') \leq \ell(E) + \ell(W) + (1+\varepsilon)\ell(F)$
For $i = 1, \ldots, k+1$ if $P_i \neq \emptyset$, let $r_i$ be the first vertex on $P_i$ such that there is an $r_i$-to-$N$ path in $F'$ that avoids other vertices of $P_i \cup \ldots \cup P_{k+1}$.

Claim: For any vertex $x$ of $P_i \setminus \{r_i, \ldots, J\}$, there is no non-trivial $x$-to-$S$ path that avoids other vertices of $P_i \cup \ldots \cup P_{k+1}$.

Proof: by picture.

Note: $r_i$ could be on $S$ in which case there is a trivial path.

For $i = 1, \ldots, k$ if there is a path in $F'$ from $P_i$ to $P_{i+1} \cup \ldots \cup P_{k+1}$, whose internal vertices are not in $P_i \cup \ldots \cup P_{k+1}$, let $Q_i$ be such a path. Otherwise let $Q_i = \emptyset$.

Claim: Every internal vertex of $Q_i$ has degree 2.

Proof: by picture.

Claim: If there is an integer $j < k$ such that $Q_j$ connects to $P_i$, then $\text{end}(Q_j) = r_i$.

Proof: by picture.
Now we can basically do. Consider each connected component \( K \) of \( F' \cup Q_2 \).

Apply the transformation shown under "Main Idea" to \( K \) to obtain \( \tilde{K} \) with few joining vertices.

- get rid of components that have leaves only on \( N \) or only on \( S \) by applying \( \text{Span} \).

One more issue: If \( r_i \) is on \( S_i \), then the northern part still needs to span any vertices of \( S_i \). Instead, we simply include the part of \( S_i \) that comes after \( r_i \) in the selection — it has small weight!

Define \( \tilde{F} = UK \tilde{V} \cup U_2 Q_2 \).

Need to verify bounds and covering requirements (F1), (F2), (F3) of Theorem. — easy! see book/paper.
References


See also upcoming chapter in Klein's book.