6.889	Lecture 8	Oct. 5, 2011

In today's lecture we will introduce branch decomposition and carving decomposition. The two are similar to tree decomposition, but sometimes easier to work with (they will probably come up later in the course, when we talk about approx. TSP.

We have two goals for today:

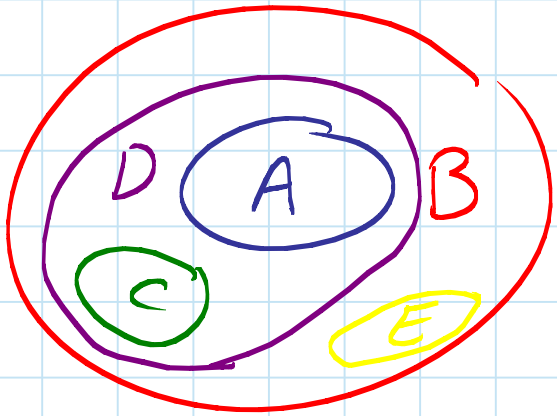
- bound the tree/branch/carving width of any planar graph G by the radius of G (or of the dual G^*)

- develope, based on the above, a framework for approximation schemes for planar graphs

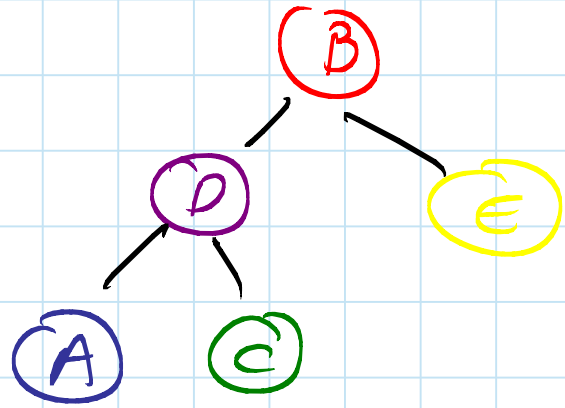
<u>Defn</u>: Two non-empty sets A, B <u>cross</u> if they are neither disjoint nor nested, i.e.

$$A \cap B \neq \emptyset \text{ and } A \not\subseteq B \text{ and } B \not\subseteq A$$

A family C of non-empty sets is <u>non-crossing</u> (also <u>laminar</u>) if no two sets in C are crossing.



non-crossing family



corresponding tree (forest)

A non-crossing family can be represented by a rooted forest under the subset relation.

For a ground set S, a <u>carving</u> of S is a maximal non-crossing family C of subsets of S.

The forest corresponding to a carving C of S is a rooted binary tree.

For a graph $G=(V,E)$, a carving of V is called a <u>carving decomposition of G</u>.

For a subset $X \subseteq V$,
let $\delta_G(X)$ denote the cut in G corresponding to X, i.e. the set of edges of G with exactly one endpoint in X.

The <u>width</u> of a carving decomposition $\mathcal{C}$ is

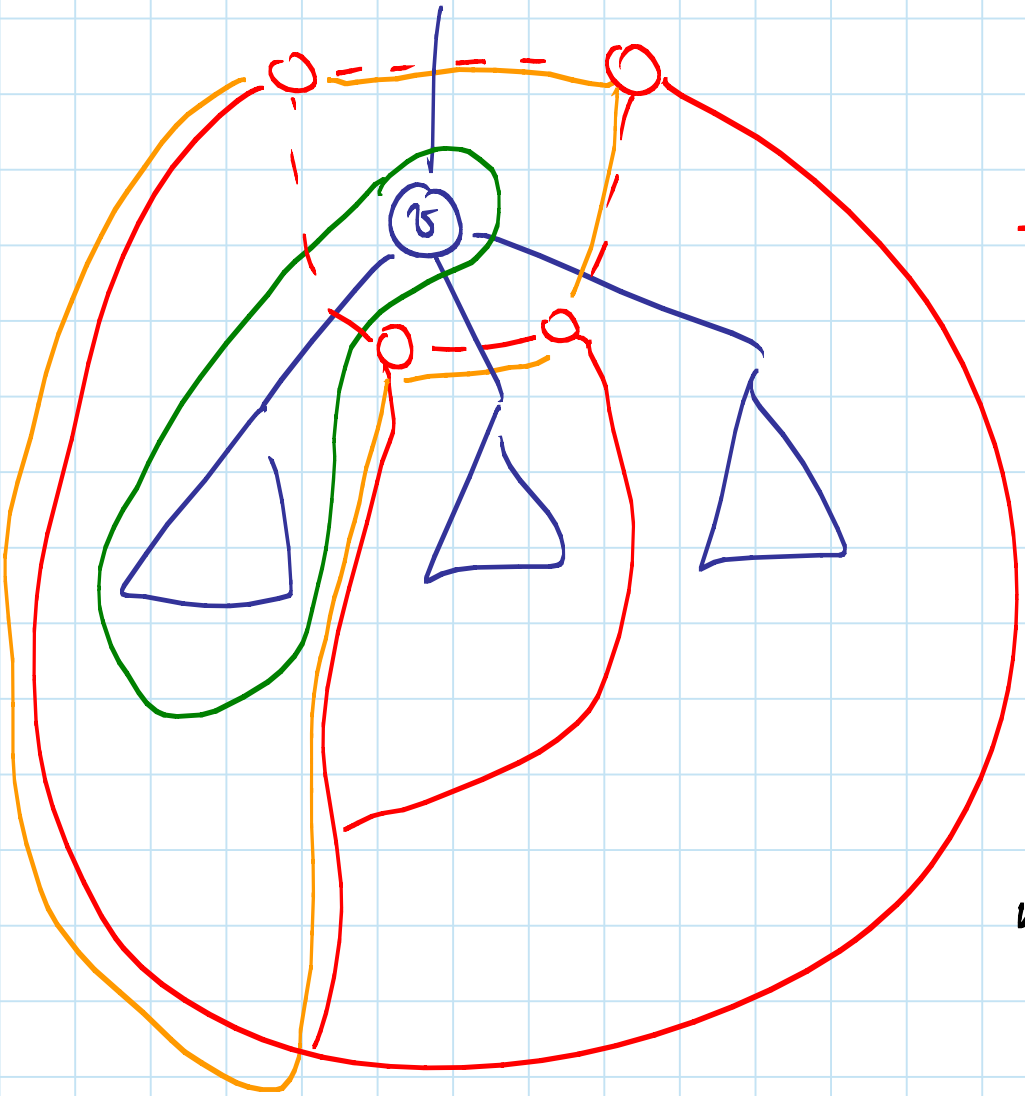
$$\max\{|\delta_G(X)| : X \in \mathcal{C}\}$$

Similar to what we've seen last lecture, many problems can be solved by dynamic programming on the carving decomposition tree. There is a DP table element for each subset of $\delta_G(X)$ for each $X \in \mathcal{C}$

The <u>carving-width</u> of G is the minimum width of a carving decomp. of G.

<u>Lemma</u>: Let $G=(V,E)$ be a planar graph of degree at most Δ. Let T^* be a spanning tree of G^*. Suppose the depth of T^* is k. Then the carving width of G is at most $2k+\Delta-1$.

<u>Proof</u>: Let T be the spanning tree of G consisting of edges not in T^*. Root T at a node r with one incident edge of T. Let $C(v)$ denote the set of descendants of v in T.
First observe that $C(r)=V$, so $\delta_G(C(r))=\emptyset$
Next, for $v \neq r$, $\delta_G(C(v))$ is the dual of the fundamental cycle of the parent edge of v w.r.t. T^*



- part of T in blue
- part of T^* in red
- the set $C_1(v)$ in green
- the dual of $\delta_G(C_1(v))$ in orange

there are at most $\Delta-1$ orange edges not in T^*

So $|\delta_G(C(v))| \leq 2k+1$. However, the set $\{C_v\}_{v \in V}$ is not maximal.

For a vertex v, let $v_1, \dots, v_s$ be its children define

$$C_i(v) = \{v\} \cup \bigcup_{j=1}^{i} C(v_i)$$

It is easy to see that the union of $C_i(v)$ over all $v \in V$ is a carving decomposition of G. By picture, width is at most $2k+\Delta-1$ $\square$

A carving C of the edge-set E is called a <u>branch decomposition</u>.

for a subset $X \subset E$, the boundary of X is $\partial_G(X)$, the set of vertices in G with some incident edge in X and some incident edge not in X.

The <u>width</u> of a branch decomposition C is

$$\max \{ |\partial_G(X)| : X \in C \}.$$

The <u>branch width</u> of G is the minimum width of a branch decomposition of G.

<u>Thm.</u> [Tamaki] There is a linear time algo. that, given a planar graph G, returns a branch-decomposition whose width is at most $3 + 2 \min \{ \text{radius of } G, \text{radius of } G^* \}$

We will prove this later in the lecture. First, let's see how to use this to get approximation schemes.

Unrelated facts:

- unlike tree width, branch width is known to be computable exactly in poly time for planar graphs [Seymour & Thomas '94]

- $\underbrace{b-1}_{\text{branch width}} \leq \underbrace{k}_{\text{tree width}} \leq \left\lfloor \frac{3b}{2} \right\rfloor - 1$

Baker's Technique

general technique. We'll use Vertex Cover as an example.

input: G output: $S \subseteq V$ with min. cardinality s.t. every edge is incident to S.

Algo: do BFS on G

$k = \frac{1}{\varepsilon}$

i: shift
j: slice

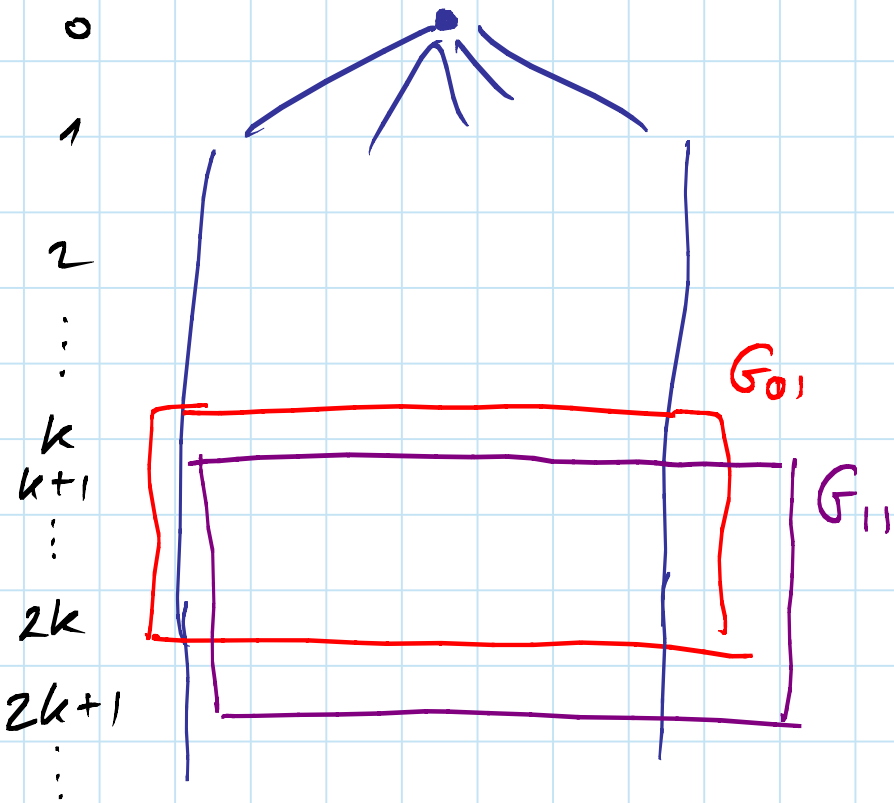
G_{ij} = subgraph induced on vertices at levels $(jk+i)$ through $(j+1)k+i$

$0 \leq i < k$

S_{ij} = min VC of G_{ij} (using DP)

$S_i = \bigcup_j S_{ij}$

return S_i with min. cardinality



<u>Running time:</u> using branch-width (or tree width) can find min VC of G_{ij} in $2^{O(k)} \cdot |V(G_{ij})|$ time.

summing over i,j we get $k \cdot 2^{O(k)} \cdot n = 2^{O(\frac{1}{\varepsilon})} \cdot n$

(aka efficient PTAS)

<u>Analysis:</u>

let OPT_i denote $OPT \cap \{\text{vertices at levels } i \bmod k\}$

OPT_i are disjoint, $\bigcup_i OPT_i = OPT$, so

$$\exists q : |OPT_q| \leq \frac{1}{k} |OPT| = \varepsilon |OPT|$$

S_{qj} is a min VC for G_{qj}

$$|S_q| \leq \sum_j |S_{qj}| \leq \sum_j |OPT \cap V(G_{qj})| = (*)$$

Consider the graphs $\{G_{qj}\}_j$. Each vertex appears once, except those at levels $q \bmod k$, which appear twice.

Hence,

$$(*) = |OPT| + |OPT_q|$$

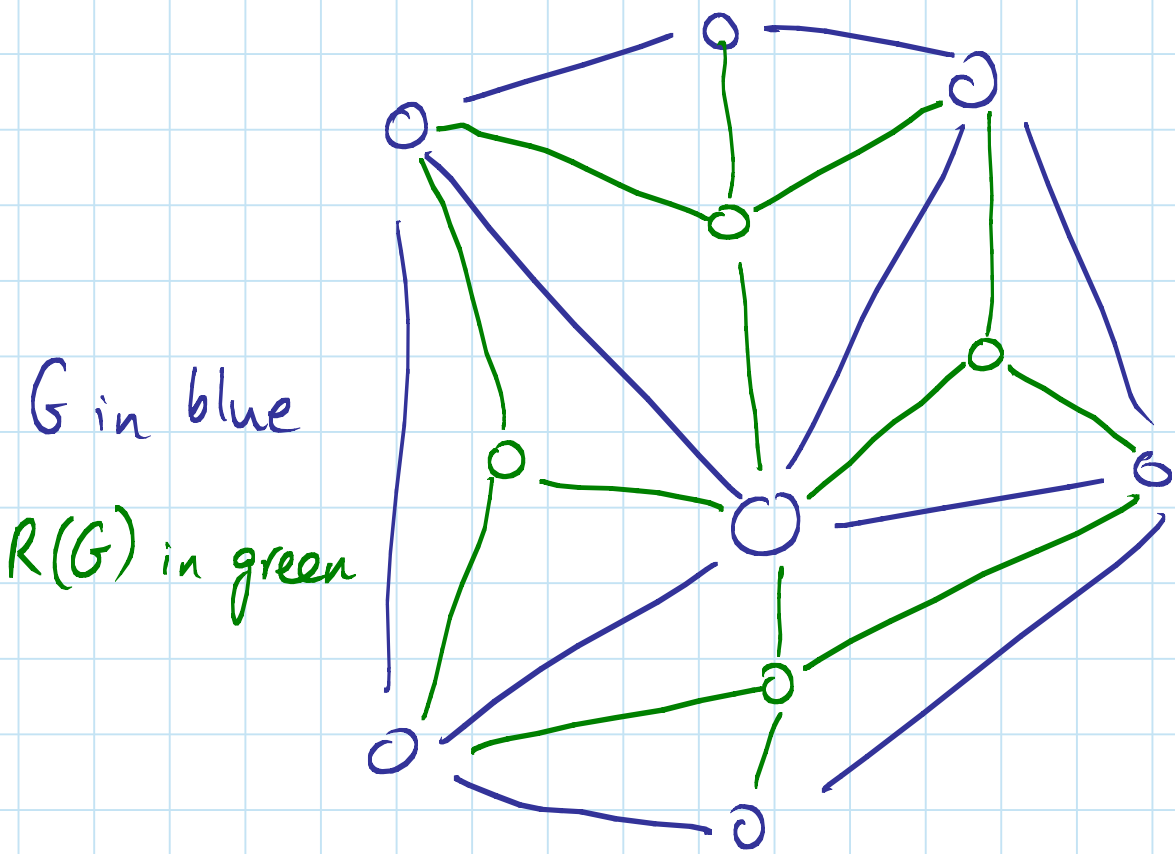
$$\leq (1+\varepsilon)|OPT|$$

The same approach can be used to devise approximation schemes for many other problems (e.g. max independant set, min dominating set)

Back to the proof of Tamaki's thm.

For an embedded graph $G=(V,E)$ with face set F, the <u>radial graph</u> $R(G)$ (also called the <u>face-vertex incidence graph</u>) is the graph whose vertices are $V \cup F$ and that has an edge vf if $v \in V$ is a vertex on $f \in F$ in G.

The <u>medial graph</u> of G, $M(G)$, is the dual of $R(G)$.



Observe:

1) $R(G)$ has the same genus as G
(in particular, $R(G)$ is planar if G is)

2) Each face of $R(G)$ has size 4
(the degree of every vertex of $M(G)$ is 4)

3) There is a bijection $b(\cdot)$ from the edges of G to the vertices of $M(G)$

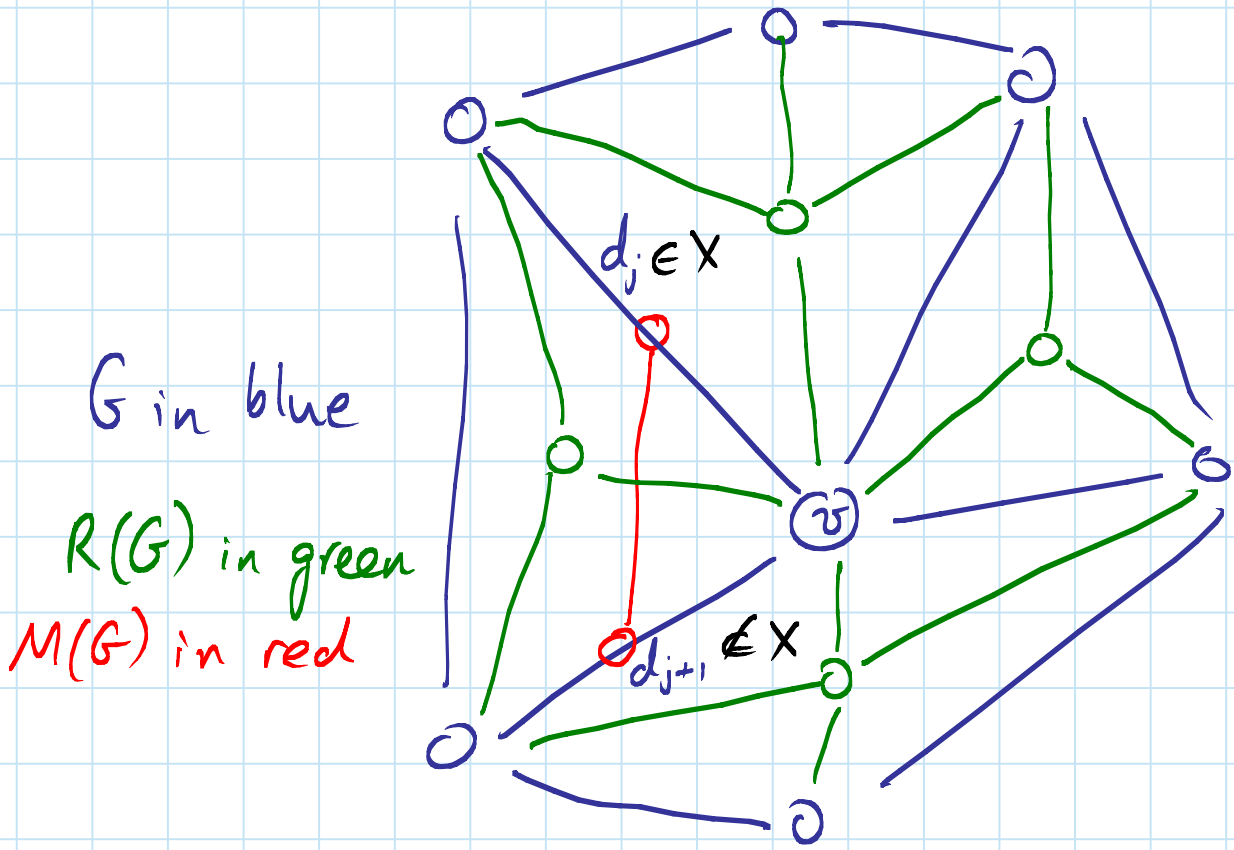
By carving width Lemma, we can construct in linear time a carving decomposition C of $M(G)$ whose width is at most $2r+4-1$, where r is the radius of $R(G)$.

Use the bijection b to get $C' = \{ b^{-1}(X) : X \in C \}$.
Easy to see that C' is a branch decomposition of G.

What is the width of C'?

Let X be a set in C'. Let $Y = b(X)$. what is $|\partial_G(X)|$?
Consider a node $v \in \partial_G(X)$. Let $v = (d_0, d_1, \dots, d_{n-1})$ and let $d_i, \dots, d_j$ be a maximal consecutive subsequence of darts of edges in X.

d_{i-1} and d_{j+1} do not belong to X.



$\Rightarrow$ $b(d_j) \in Y$, $b(d_{j+1}) \notin Y$ so edge of $M(G)$ whose endpoints are $b(d_j)$ and $b(d_{j+1})$ is in $\delta_{M(G)}(Y)$. Associate this edge with v.

same for d_i and d_{i-1}.
We see that there are at least two edges of $\delta_{M(G)}(Y)$ associated with v.

Hence $$|\partial_G(X)| = \frac{1}{2} \cdot \sum_{v \in \partial_G(X)} 2 \leq \frac{1}{2} \sum_{v \in \partial_G(X)} \left(\begin{array}{c} \# \text{ of edges of} \\ \delta_{M(G)}(Y) \text{ assoc.} \\ \text{with } v \end{array} \right)$$

$$\leq \frac{1}{2} |\delta_{M(G)}(Y)| = \frac{1}{2}(2r+3)$$

so the branch width of G is bounded by $r+1$.

Recall that r is radius of $R(G)$

For every u-to-v path in G with x edges there is a u-to-v path in $R(G)$ with $2x$ edges.
Same for G^*.

So radius of $R(G)$ is at most

$$r \leq 2 + 2\min\{\text{radius of } G, \text{radius of } G^*\}$$

So branch-width $\leq 3 + 2\min\{\text{radius of } G, \text{radius of } G^*\}$

$\square$