6.889 — Lecture 3: Planar Separators

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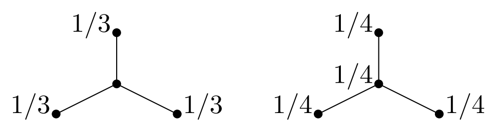
We shall prove theorems of the following flavor (see textbook/papers for precise statements and proofs).

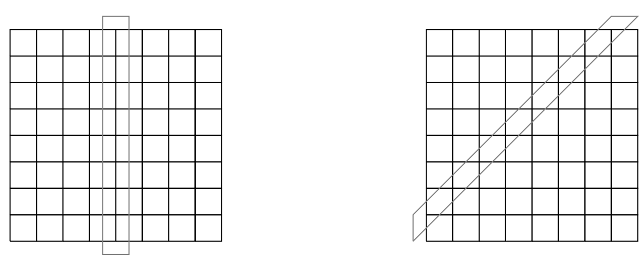
Thm. *For any planar graph $G = (V, E)$ on $n = |V|$ vertices and for any*[1] *weight function $w : V \to \mathbb{R}^+$, we can partition V into $A, B, S \subseteq V$ such that*

- [α–balanced] $w(A), w(B) \leq \alpha \cdot w(V)$ *for some* $\alpha \in (0, 1)$
- [separation] *no edge between any $a \in A$ and $b \in B$ ($A \times B \cap E = \emptyset$)*
- [small separator] $|S| \leq f(n)$
- [efficient] A, B, S *can be found in linear time.*

Trees what if G is a (binary) tree? can do $1/2$–balanced partition with $|S| = 1$? $\rightsquigarrow$ only $2/3$–balanced! with one *edge* in separator? $\rightsquigarrow$ only $3/4$–balanced for binary trees



Grids What happens for a grid on n vertices (say square: $\sqrt{n} \times \sqrt{n}$)? $|S| \leq \sqrt{n}$



$1/2$–balanced $\rightsquigarrow \sqrt{n}$ $\qquad$ $2/3$–balanced $\rightsquigarrow < \sqrt{n}$ but $\Theta(\sqrt{n})$

cut out a diagonal and remain $2/3$–balanced, s vertices separate $\approx s^2/2$ vertices from the rest, $n/3 \leq s^2/2$. $\mathcal{O}(\sqrt{n})$ is "right order of magnitude" $\rightsquigarrow$ today's lecture: can generalize to *all* planar graphs

Beyond Extensions to bounded-genus and minor-free graphs to be discussed in Lecture 5

General Graphs Is there a separator theorem that works for *any* graph? No (complete graph)! For any sparse graph? No (expander graphs)!

[1] almost — need individual weights $\leq (1 - \alpha)w(V)$

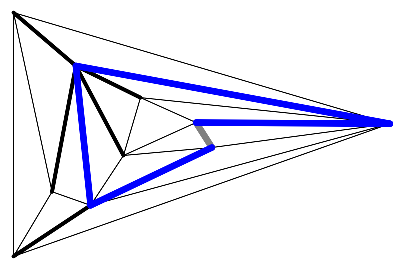
1 Fundamental Cycle Separator

We shall prove two versions of the main theorem. Both proofs use the following lemma (a weighted version).

Lemma. *For any planar graph $G = (V, E)$ with a spanning tree of radius d rooted at $r \in V$, we can partition V into $A, B, S \subseteq V$ such that*

- [balanced] $|A|, |B| \leq 3n/4$
- [separation] *no edge between any $a \in A$ and $b \in B$ ($A \times B \cap E = \emptyset$)*
- [separator size] $|S| \leq 2d + 1$
- [efficient] A, B, S *can be found in linear time.*

Proof (sketch). Let T be the spanning tree of depth d rooted at r. Triangulate G. Recall *interdigitating trees* from Lecture 2. Let T^* be the dual tree in the triangulated version of G. Every non-tree edge e defines a *fundamental cycle* $C(e)$. Since T has depth d, we have $|C(e)| \leq 2d + 1$.
$\rightsquigarrow$ assign appropriate weights to faces. then find edge separator in interdigitating tree! (T^* has degree 3)



Problem Set: how to *efficiently* find the best edge e (how to compute $w(\text{ext}(C(e))), w(\text{int}(C(e)))$ for each edge e, where $\text{ext}(C), \text{int}(C)$ denote the *exterior* and *interior* of a cycle C, respectively) □

Problem Diameter of G may be large! Want $|S| \leq \sqrt{n}$. $\rightsquigarrow$ two ways to reduce the diameter

2 Vertex Separators

We prove the theorem with $|S| \leq 4\sqrt{n} + \mathcal{O}(1)$. Algorithms with better constants are known.

Overview *1)* Diameter Reduction in primal $\rightsquigarrow G'$ *2)* Fundamental Cycle in G' (lemma)

Algorithm

- Breadth-First Search (BFS) from any $v \in V$, let $L_i(v)$ denote all the vertices at level i of the BFS tree
  Note: any level $L_i(v)$ is a separator — not necessarily balanced, not necessarily small
  Define sentinel level $L_{\Delta+1}(v) = \emptyset$, where Δ denotes diameter of G
- Find level i_0 with the median vertex ($\sum_{i \leq i_0} |L_i(v)| \geq n/2$ and $\sum_{i \geq i_0} |L_i(v)| \geq n/2$)
- Find levels $i_- \leq i_0 \leq i_+$ (start from i_0 and decrease i_- / increase i_+) until $|L_{i_-}|, |L_{i_+}| \leq \sqrt{n}$. by counting argument (each part has only half the vertices), we have that $|i_0 - i_-|, |i_+ - i_0| \leq \sqrt{n}/2$.

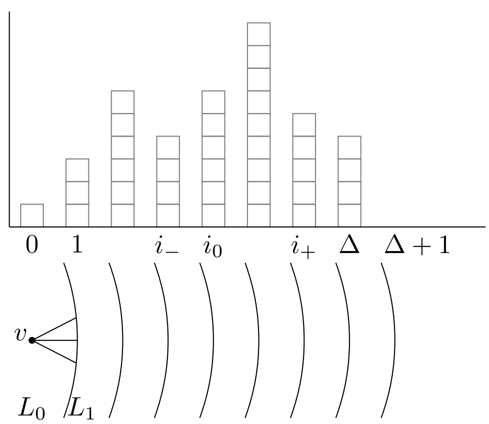
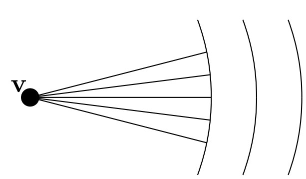


Figure 1: Levels of the breadth-first search tree rooted at v and vertex count per level.

  have separator $|L_{i_-} \cup L_{i_+}| \leq 2\sqrt{n}$; Return if some combination of $L_{<i_-}, L_{>i_+}, L_{(i_-,i_+)}$ is balanced
- Heavy part is in $L_{(i_-,i_+)}$. Why? (median!) to apply the lemma, form a graph G' as follows:
  - delete (contraction also works) $L_{\geq i_+}$
  - contract all edges in $L_{\leq i_-} \rightsquigarrow$ super vertex $\mathbf{v}$, connected to all $u \in L_{i_-+1}$



  BFS tree in G' rooted at $\mathbf{v}$ has depth $|i_+ - i_-| \leq \sqrt{n}$, triangulate, apply lemma, let C denote the cycle
- Return some combination of int(C), ext(C), $L_{<i_-}, L_{>i_+}$ as A and B and some combination of C and L_{i_-}, L_{i_+} as separator S

3 Recursive Separation

We apply the theorem recursively to obtain an r–division.

Def. *An* r–division *of G is a decomposition into*

- $\mathcal{O}(n/r)$ *edge-disjoint* pieces*,*
- *each with $\leq r$ vertices and*
- $\mathcal{O}(\sqrt{r})$ boundary *vertices.* $\Leftarrow$ *vertices with edges to at least two pieces*

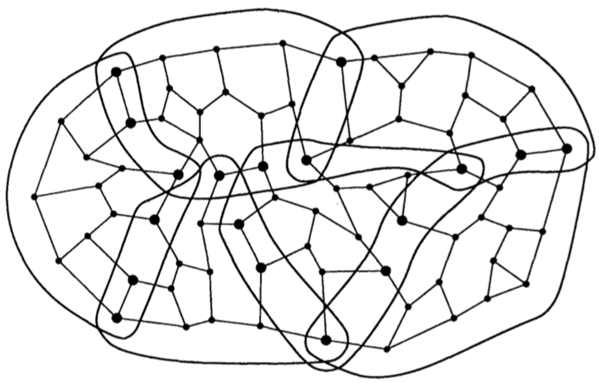


Figure 2: Illustration of an r–division, extracted from [Fre87, p. 1006, Fig. 1]

Lemma. *For planar G, we can compute an r–division in time $\mathcal{O}(n \log n)$.*

Proof (sketch). Two phases.

1) Total Boundary $\mathcal{O}(n/\sqrt{r})$. apply thm $\rightsquigarrow$ two pieces $A' \subseteq A \cup S, B' \subseteq B \cup S$, sizes $\alpha n + \mathcal{O}(\sqrt{n})$ and $(1-\alpha)n + \mathcal{O}(\sqrt{n})$ with $\alpha \in [1/4, 3/4]$. let $B(n,r)$ denote the number of boundary vertices. recurrence: $B(n,r) = 0$ for $n \leq r$ and

$$B(n,r) \leq \mathcal{O}(\sqrt{n}) + B(\alpha n + \mathcal{O}(\sqrt{n}), r) + B((1-\alpha)n + \mathcal{O}(\sqrt{n}), r) \quad \text{for } n > r$$

2) $\mathcal{O}(\sqrt{r})$ Boundary per Piece. WHILE there is piece P with large boundary of size n', apply thm to P with weights s.t. boundary vertex has weight $1/n'$ and interior vertex weight $0 \rightsquigarrow$ separates boundary vertices prove that number of pieces and total boundary still bounded (details in textbook and papers) □

4 Example Application: Divide & Conquer for Planar Graphs

MAXIMUM INDEPENDENT SET (MIS) Problem: find set of maximum size $I \subseteq V$ with no two vertices adjacent, classical **NP**–complete problem, also hard to approximate (**MaxSNP**–complete)

Approximation Algorithm recursively apply separator theorem until separated sets (*pieces*) have size $\log\log n$; find MIS $I(P)$ per piece P (by exhaustive search, $\mathcal{O}(2^{\log\log n})$ per piece); return union $\bigcup_P I(P)$

Analysis total number of boundary vertices is $\mathcal{O}(n/\sqrt{\log\log n})$. let I^* denote optimal solution. have $|I(P)| \geq |I^*(P)|$ for each piece P. Thus $|I^*| - |I| \leq \mathcal{O}(n/\sqrt{\log\log n})$. Planar graphs are 4–colorable, which implies $|I^*| \geq n/4$. therefore, relative error is at most $\frac{|I^*|-|I|}{|I^*|} = \mathcal{O}(1/\sqrt{\log\log n})$.

5 Cycle Separators

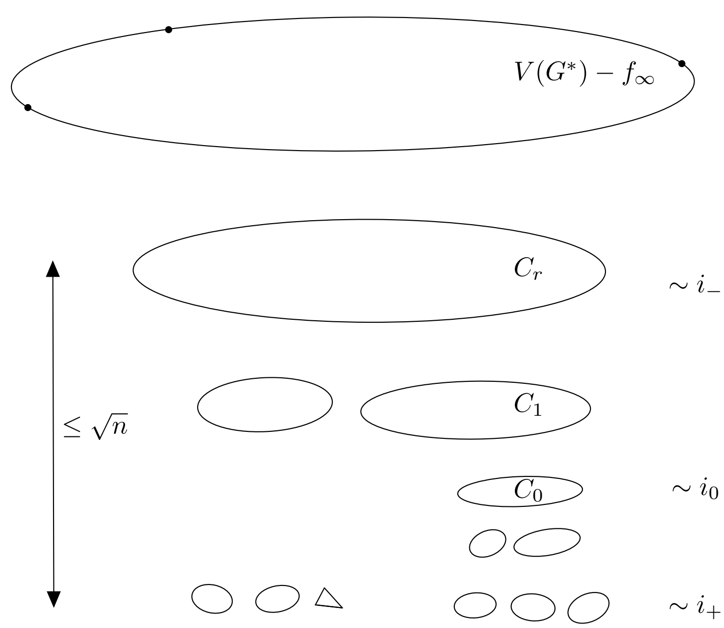
Problem vertex separator S is small but not very "nice," problem in some applications, simple cut desired

Solution prove thm for S a *cycle*. what if G is a tree? one triangle does not help either. need 2–connectivity! what if G is a cycle itself? separator size depends on face sizes! for this lecture: assume G is triangulated.

Idea and Overview reduce diameter to $\mathcal{O}(\sqrt{n})$ *without* having to add nodes/edges to separator. *merge* faces without making the resulting face too heavy (weight $> 1/2$) or too big (# nodes $> \sqrt{n}$)
1) Diameter Reduction in dual ["almost" above arguments for G^*] $\rightsquigarrow$ G' *2)* Fundamental Cycle in G'

Algorithm (sketch)

- BFS in the *dual* G^*, rooted at any face f_∞. as above, let $L_i(f_\infty)$ denote all the vertices (here: faces) at level i of the BFS tree. front of the search is collection of cycles. union of their *exteriors* is explored; union of their *interiors* is yet unexplored. each cycle C has *weight* (interior of C, yet unexplored) and *boundary* (cycle length $|C|$)
- compute heavy subtree of BFS tree as follows:
  - start at root, DO follow heaviest child (cycle weight) UNTIL reach cycle C_0 with weight $> 1/2$ and all enclosed cycles have weight $\leq 1/2$ ("BFS-deepest heaviest cycle").
  - find level i_- with boundary size $\mathcal{O}(\sqrt{n})$ (above counting arguments). within L_{i-} choose cycle C_r enclosing C_0.
  - find level i_+ with small total boundary (at most $\mathcal{O}(\sqrt{n})$)



- obtain low-diameter *primal* graph G': for cycles C in L_+ enclosed by C_0 *merge* faces enclosed by C (contract edges in the dual $\Leftrightarrow$ delete edges in the primal); merge faces *not* enclosed by C_r. G' has diameter $\mathcal{O}(\sqrt{n})$ (diameter in dual, triangulated, plus merged faces)
- compute spanning tree T' (*almost* BFS) and apply lemma to G' with T'

References

Separators Ungar [Ung51] proved the existence of separators of size $\mathcal{O}(\sqrt{n \log n})$. Lipton and Tarjan [LT79] gave a linear-time algorithm to find a $2/3$–balanced separator of size $\sqrt{8n}$. Many applications (such as the MIS approximation algorithm) can be found in the companion paper [LT80]. Djidjev [Dji82] improved the bound on the separator size to $\sqrt{6n}$. He also proved a lower bound of $(\sqrt{4\pi\sqrt{3}}/3) \cdot \sqrt{n}$. Variants of these algorithms have been implemented and evaluated experimentally [HSW+09].

Miller [Mil86] gave a linear-time algorithm to construct a cycle separator of size $\sqrt{8n}$ for 2–connected, triangulated planar graphs. Djidjev and Venkatesan [DV97] improved the bound on the cycle length to $2\sqrt{n}$. Results extend to planar graphs with larger faces, introducing a multiplicative dependency on the root of the largest face size d (see [Mil86]), or an additive dependency on the ℓ_2–norm of all face sizes [GM90]. A rich body of further work investigates different balance criteria and many other interesting questions.

Recursive Separators Frederickson [Fre87] gave an algorithm that computes an r–division in $\mathcal{O}(n \log r)$ time. Goodrich [Goo95] gave a linear-time algorithm.

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[Ung51] Peter Ungar. A theorem on planar graphs. *Journal of the London Mathematical Society*, s1-26(4):256–262, 1951.