

3 ways to interpret graphs:

- directed (arcs)
- undirected (edges)
- bidirected (darts)

Let A be a finite set (arc set)

the set of darts is $A \times \{+1, -1\}$

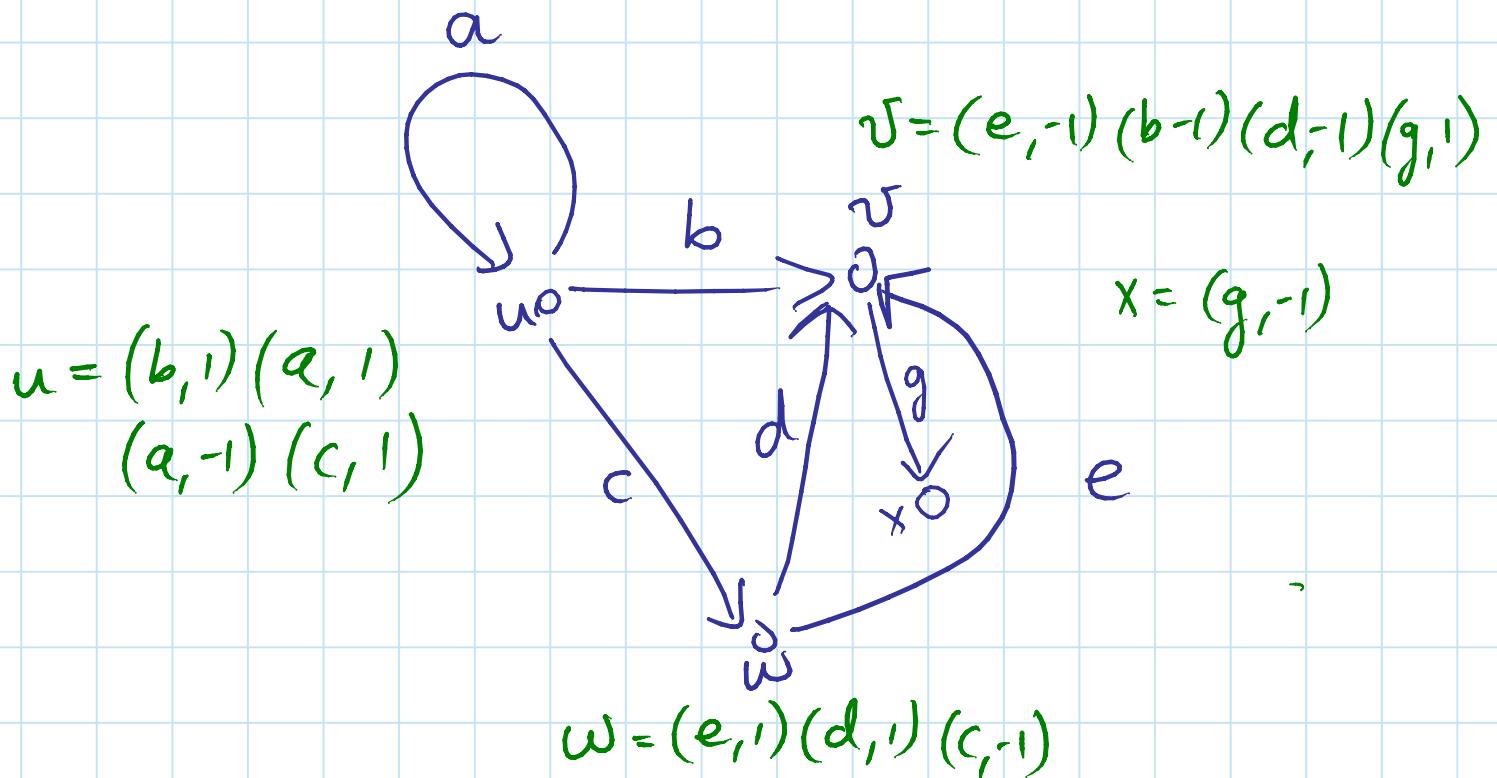
$(a, 1)$ is often identified with the arc a .

A graph G is a pair (V, A) where V is a partition of the dart set. That is, nodes are defined by their outgoing darts

the tail of arc a is the node $v \in V$ to which $(a, 1)$ belongs.

the head of arc a is the node $v \in V$ to which $(a, -1)$ belongs.

We will sometimes denote arcs by their endpoints
e.g. $a = uv$ where $u(v)$ is a 's tail (head)



note that our definition does not allow isolated nodes

define the bijection $\text{rev}((e, \sigma)) = (e, -\sigma)$

the graph obtained from (V, A) by deleting an edge set A' is $(V', A - A')$, where V' is the restriction of V to the darts in $A - A'$

contracting an edge $uv \in A$ from (V, A) produces the graph (V', A') where $A' = A - \{uv\}$ and the parts u, v of V are merged (the darts of uv are removed)

Think about making a contracted edge shorter and shorter until its endpoints meet. **Not well defined for self loops!**

Embeddings: We will use combinatorial embeddings (aka rotation system)

An embedding of $G = (V, A)$ is a permutation π on $A \times \{+1, -1\}$ whose orbits (cycles) are exactly the nodes (parts) of V .

[think of π as specifying, for every $v \in V$, the darts whose tail is v in, say, counterclockwise order.]

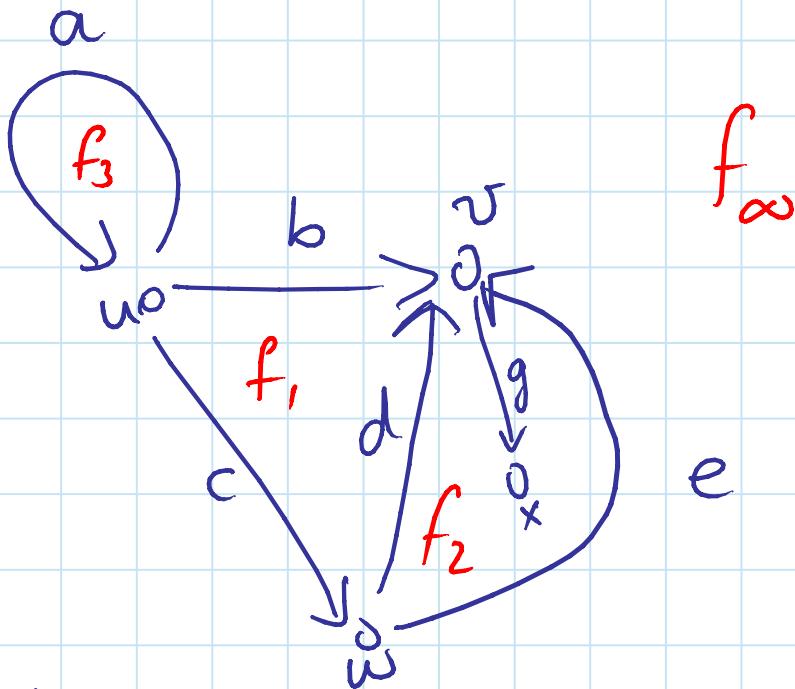
An embedded graph is the pair $G = (\pi, A)$
we will also use the notation G_π

Faces: define $\pi^* = \pi \circ \text{rev}$

the faces of $G = (\pi, A)$ are the orbits of π^*

[When working with topological embeddings the faces of G are the connected components of the set of points in the sphere that are not assigned to any node or edge.]

e.g. :



$$\pi = \underbrace{\left((a_{-1})(c_1)(b_1)(a_1) \right)}_{\omega} \underbrace{\left((e_{-1})(b_{-1})(d_{-1})(g_1) \right)}_{x} \\ \underbrace{\left((c_1)(d_1)(c_{-1}) \right)}_{\omega} \quad \underbrace{\left((g_{-1}) \right)}_{x}$$

$$\pi^* = \underbrace{\left((b_1)(d_{-1})(c_{-1}) \right)}_{f_1} \underbrace{\left((d_1)(g_1)(g_{-1})(e_{-1}) \right)}_{f_2} \\ \underbrace{\left((e_1)(b_{-1})(a_1)(c_1) \right)}_{\underbrace{f_\infty}_{f_3}} \quad \underbrace{\left((a_{-1}) \right)}_{f_3}$$

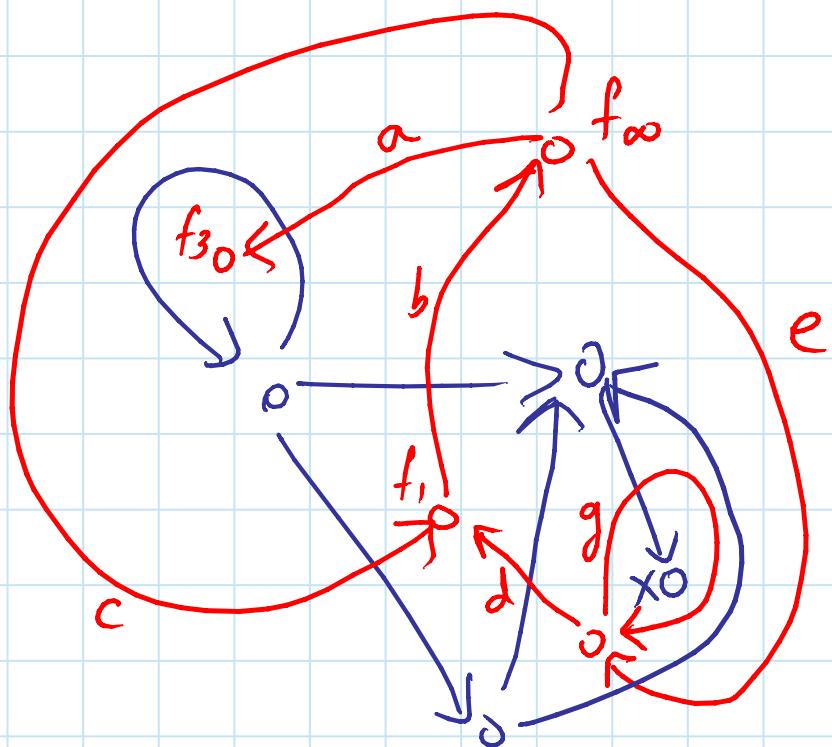
Note: in our drawing, every face except f_∞ corresponds to a clockwise simple cycle. f_∞ is called the infinite face. combinatorial embeddings do not distinguish infinite face (think of embedding on a sphere).

note: with combinatorial embeddings each connected component has its own infinite face

note: for connected graphs, combinatorial embeddings are equivalent to topological embeddings

Dual graph: the dual of $G = (\pi, A)$ is

the embedded graph $G^* = (\pi^*, A)$



note: when drawing G^* on G , the order of darts in π^* corresponds to clockwise order around dual nodes.

"look at G^* from the other side of the paper"

$$\begin{aligned}\pi^* = & \left((b,1) (d,-1) (c,-1) \right) \left((d,1) (g,1) (g,-1) (e,-1) \right) \\ & \left((e,1) (b,-1) (a,1) (c,1) \right) \left((a,-1) \right)\end{aligned}$$

Lemma: the dual of the dual is the primal
 $(G^*)^* = G$

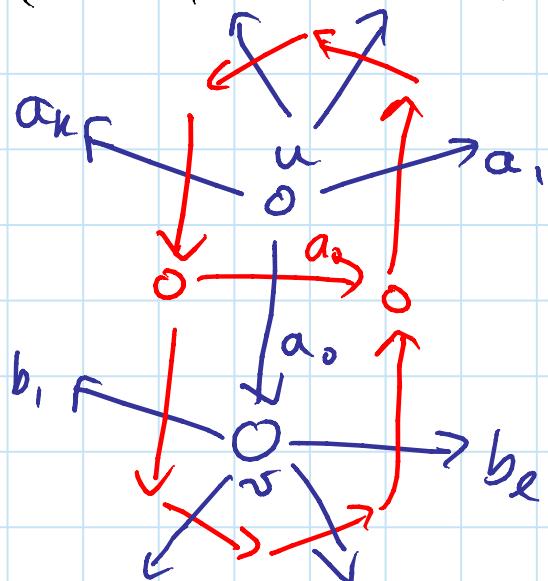
Proof: $(\pi^*)^* = (\pi \circ \text{rev}) \circ \text{rev} = \pi$ \square

How do combinatorial embeddings behave under deletions and contractions?

Deleting a dart d from G_π creates $G_{\pi'}$ where

$$\pi'(d') = \begin{cases} \pi \circ \pi(d') & \text{if } \pi(d') = d \\ \pi(d') & \text{otherwise} \end{cases}$$

Lemma: contracting an edge of G_π is equivalent to deleting it in the dual G^*
 (recall contraction is not defined for self loops)



intuition: in G^* , u and v are faces. deleting the edge uv in G^* merges these two faces, so in G u and v are merged.

Proof: Let uv be the edge to be deleted. Let (a_0, a_1, \dots, a_k) and $(b_0, b_1, \dots, b_\ell)$ be the orbits of π that correspond to u and v , respectively, and such that a_0 and b_0 are the darts of uv . Since uv is not a self loop the two orbits are distinct. We want to show that $(\pi^{*'})^*$ is identical to π except that these two orbits are merged into $(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_\ell)$

$$(\pi^{*'})^*[d] = \pi^{*'} \circ \text{rev}(d)$$

$$= \begin{cases} \pi^{*'}[\pi^{*'}[\text{rev}(d)]] & \text{if } \pi^{*'}[\text{rev}(d)] \text{ is deleted} \\ \pi^{*'}[\text{rev}(d)] = \pi(d) & \text{otherwise} \end{cases}$$

now, $\pi^{*'}[\text{rev}(a_k)] = \pi \circ \text{rev} \circ \text{rev}(a_k) = a_0$
 and $\pi^{*'}[\text{rev}(b_\ell)] = \pi \circ \text{rev} \circ \text{rev}(b_\ell) = b_0$

so a_k, b_ℓ are the only two darts s.t. $\pi^{*'}[\text{rev}(d)]$ is deleted. Hence

$$(\pi^{*'})^*[d] = \begin{cases} \pi^{*'}[a_0] = \pi[b_0] = b, & \text{if } d = a_k \\ \pi^{*'}[b_\ell] = \pi[a_0] = a, & \text{if } d = b_\ell \\ \pi[d] & \text{otherwise} \end{cases}$$

so $(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_\ell)$ is a new orbit after the deletion in the dual. \square

Planarity, we say that an embedding π of $G = (V, A)$ is **planar** if it satisfies Euler's formula

$$n - m + f = 2k$$

of nodes # of arcs # of faces # of connected components

more generally, it can be shown that for any embedding π , $n - m + f = k(2 - 2g)$

g is the **genus** of the embedding. Planar embeddings have $g = 0$.

you will prove in the PS that.

Sparsity lemma: for any planar embedded in which every face has size at least 3,

$$m \leq 3n - 6$$

implies no
self loops &
no parallel edges

Interdigitating trees lemma: let T be a spanning tree of a planar embedded graph $G = (\pi, A)$. The edges not in T form a spanning tree of G^* .

Proof: (see draft of book for complete proof using combinatorial embeddings)

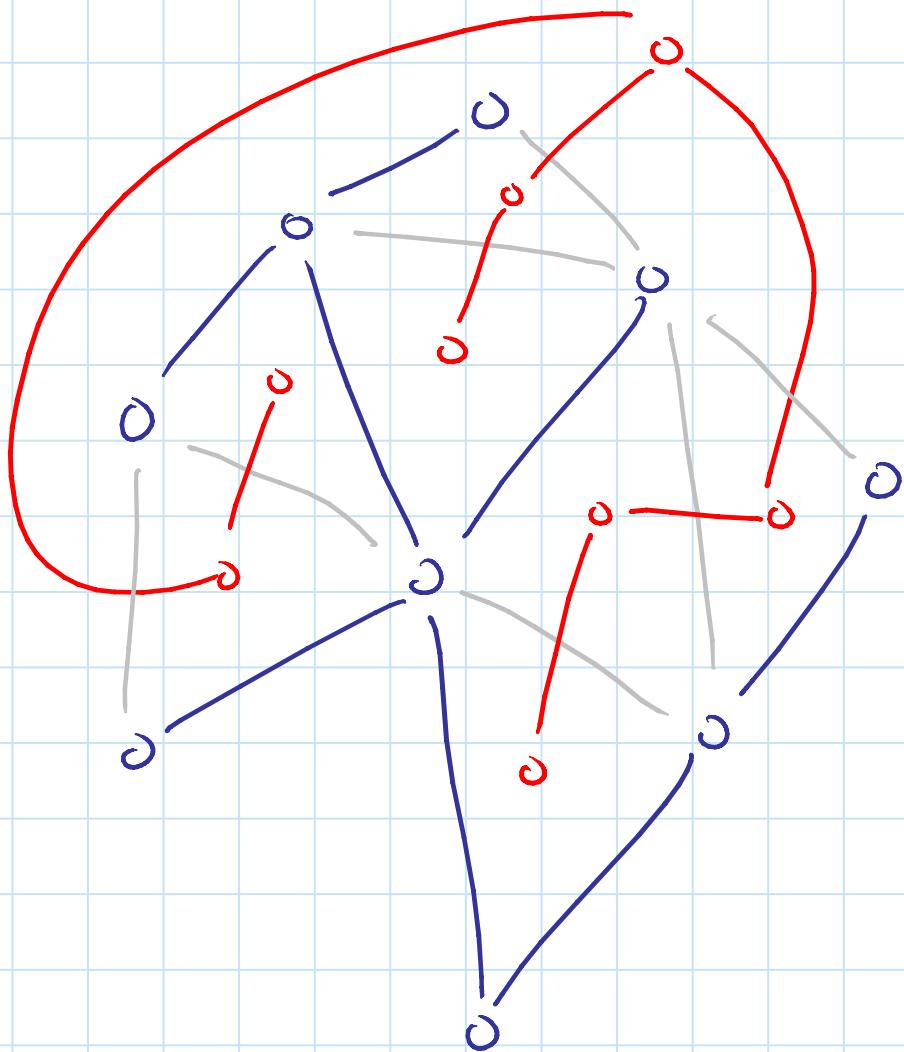
we first show that the edges not in T form a forest in G^* .

Consider a cycle C in G^* . By the Jordan curve theorem C partitions the sphere into two connected regions. Each of these regions contains at least one node of G . Hence, considered as curves in the plane T crosses C , so C contains at least one edge of T . This implies that the subgraph induced by edges not in T is acyclic in G^* , namely a forest.

It remains to show that the forest is a spanning tree. Since T is a tree, $|T| = |V| - 1$.

By Euler's formula $|V| - |A| + |V^*| = 2$

Hence $|A| - |T| = |V^*| - 1$ so the forest is indeed a spanning tree. \square



For a spanning tree T and a non-tree edge e , the fundamental cycle of e with respect to T is the cycle that consists of e and of the unique simple path in T between the endpoints of e .

Cycle-Cut duality:

Recall that for a set X of nodes, the edge cut $\delta(X)$ is called a bond (or a simple cut) if both sides of the cut are connected.

Lemma: Let $G = (\pi, A)$ be a planar graph. a set C of edges is a simple cycle in G iff it is a simple cut in G^* .

Proof: (again, see book for a proof that uses just combinatorial embeddings)

Let C be a cycle in G . Let X be the set of faces of G (nodes of G^*) enclosed by C . By the Jordan curve theorem, any path between X and $V^* \setminus X$ must cross C , so the edges of C form a cut in G^* .

For any two dual nodes $f, g \in X$ there is a curve in the sphere that connects f and g and crosses no nodes of G . It follows that f and g are connected in the restriction of G^* to X .

A symmetric argument holds for the restriction of G^* to $V^* \setminus X$.

□

Connectivity

Lemma: For any face f of any embedded graph G_π the darts comprising f are connected.

Proof: Let (d_0, d_1, \dots, d_k) be the orbit of π^* that corresponds to f . We show that d_0, d_1, \dots, d_k is a walk in G . for $1 \leq j \leq k$ $d_j = \pi^*(d_{j-1})$ so $\pi(\text{rev}(d_{j-1})) = d_j$, so d_j and $\text{rev}(d_{j-1})$ have the same tail in G , so the head of d_{j-1} is the tail of d_j . \square

Connectivity Lemma: a set of darts forms a connected component in $G = (\pi, A)$ iff the same set forms a connected component in G^* .

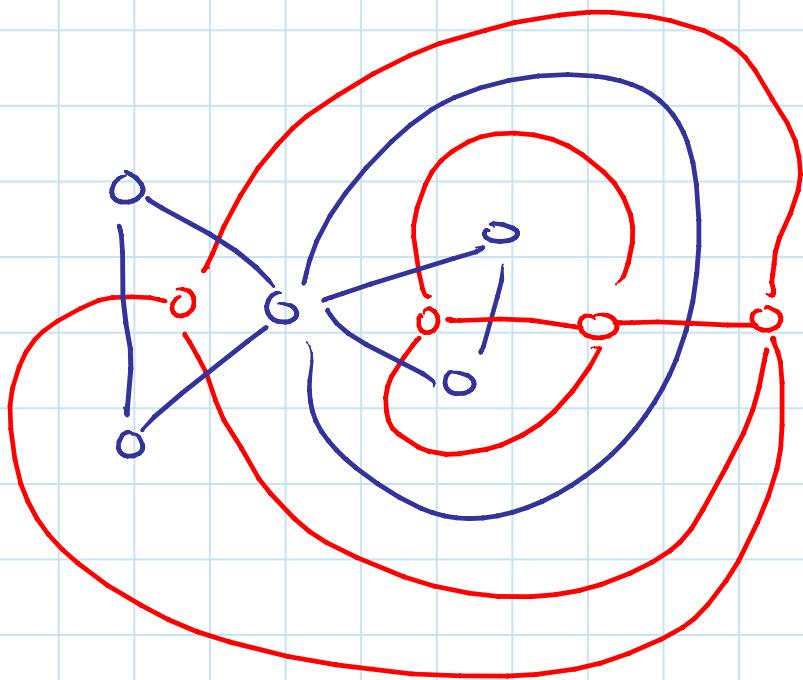
Proof: suppose d, d' are connected in G , and let

$d = d_0, d_1, d_2, \dots, d_k = d'$ be a path of darts connecting them. For $i=1, 2, \dots, k$, the head of d_{i-1} in G is the tail of d_i . Thus, d_i and $\text{rev}(d_{i-1})$ are in the same orbit of π , so are on the same face in G^* . Hence, $\text{rev}(d_{i-1})$ and d_i are connected in G^* and so are d_{i-1} and d_i .

Compression: We define compression of an edge as deleting it in the dual.

Already saw that compressing a non-self loop is contraction.

How about compressing self loops?



if e is a self loop in G then it is a cut-edge in G^* . Deleting e from G^* makes it disconnected, so makes G disconnected as well

