Locked & unlocked chains of planar shapes:
rigid objects in place of bars

- simple locked examples:
  [M. Demaine 1998]

- equilateral Δs

Adorned chain: view shapes as adornments attached to bar connecting hinges

- underlying chain linkage
- some flexibility in first & last shape

Adornment = shape + base

- shape = simply connected compact planar region
- base = line segment connecting 2 boundary pts.
- require base to be contained in shape
Slender adornment = walking along boundary from one base endpoint to the other monotonically increases distance to former (& decreases distance to latter)

= all inward normals hit the base (for piecewise-differentiable shapes)

= (possibly infinite) union of half-lenses: intersection of disks centered at base endpoints & halfplane on one side of base

⇒ can define slender hull = union of half-lenses thru every point in adornment

Slender ⇒ not locked: expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Intuition: consider two touching adornments
  - draw collinear inward normals from touching point x
  - resulting points a & b expand (vertices expand ⇒ points on bars expand)
  ⇒ two copies of x locally expand
  - in reality, this argument is tricky; can stay equal, to first order
  - possible with strict expansiveness [see SoCG 2006 proof]
Symmetric case: adornments reflectionally symmetric about their bases
⇒ slender adornment = union of lenses

Stronger result for this case:
instantaneous expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Proof: take any two lenses of different adornments - nonintersecting before the motion i.e. four disks have empty intersection

Kirszbraun's Theorem: [1934]
if we instantaneously translate n disks with an empty n-way intersection according to an expansive motion on their centers, then they still have empty intersection

(annoying detail: Kirszbraun's disks include their boundary, but our disks might kiss — but Kirszbraun's Theorem holds for disks excluding their boundaries, by taking limits of slightly smaller disks)

If initially intersecting, can show area of union only increases [Bezdek & Connelly 2002 ~Kneser-Poulsen conj.]
Proof that slender \( \Rightarrow \) not locked: (general case)

( not true for instantaneous: \( \overset{\text{desk}}{\text{chair}} \Rightarrow \overset{\text{chair}}{\text{desk}} \) )

- take any 2 half-lenses of different adornments
- consider transition time from not intersecting to intersecting \( \Rightarrow \) touching
- 3 types of touching:
  1. bases of both
     - non-intersection guaranteed by underlying chain linkage
  2. base of one
     - can add symmetric lens of other, & just consider base of first \( \times \)
     - no intersection by symmetric case
  3. base of neither
     - can add symmetric lens of both
     - again no intersection by symmetric case \( \square \)

Carpenter's rule theorem \( \Rightarrow \) straighten/convexify any slender-adorned (non-self-touching) chain
\( \Rightarrow \) connected config. space of open chains
- not true of closed chains:
**OPEN:** which adornments never lock in a chain? (like slender)

**Triangles:** not locked if angles opposite base $\geq 90^\circ$ (right or obtuse)

- locked $\approx$ identical equilateral triangles:

- can stretch/shrink in y coord. to make locked example with any angle $< 90^\circ$

**Proof:** show self-touching version rigid $\Rightarrow$ strongly locked  [Lecture 5]

- Rule 1: $\angle < 90^\circ$ $\Rightarrow$ because inner bar must stay pinned against outer bar until an angle $\geq 90^\circ \Rightarrow$ positive time

- Rule 2: $< 90^\circ$ $\Rightarrow$ by same argument
Locked triangles proof: (cont’d)

- Clearly rigid if zero-length struts were bars.
- Set \( s(AB) = -s(AB') < 0 \Rightarrow A \) in equilibrium.
- Set \( s(BC) = s(AB) = -s(B'C') = -s(AB') < 0 \)
  \( \Rightarrow \) force on \( B, B' \) vertical \( \Rightarrow \) in equilibrium if
  set \( s(B, AB') = s(B, B'C') < 0 \) appropriately.
- Set \( s(C'D'), s(D', DC), s(D', DE) < 0 \) unique up to scale
  to put \( D' \) in equilibrium; scale very small.
- Set \( s(CD) = -s(C'D') \Rightarrow D \) in equilibrium (inverse of \( D' \))
- \( s(BC) < 0 \) dominates \( s(CD) \Rightarrow \) can set \( s(C, C'B') \) &
  \( s(C, C'D') < 0 \) to put \( C \) & hence \( C' \) in equilibrium. \( \Box \)
Locked tree arguments: [some new realizations]

- Rule 1 immediately rigidifies “triangle tree”:

  \[ \Rightarrow \text{Rule 1} \]

  \[ \rightarrow \text{rigid} \]

Stress argument for Biedl et al. tree: [Connelly, Demaine, Rote 2002]

- Clearly rigid if zero-length struts were bars
- \( s(CB), s(CAO), s(CA''O) < 0 \) unique up to scale to put \( C \) in equilibrium
- \( s(BA), s(BA''O) < 0 \) uniquely determined for \( B \)
- Symmetric on petals \( \Rightarrow \) \( O \) in equilibrium
- Force so far on \( A \) must be parallel to \( A \)
  (because forces must induce zero rotation around \( O \))
- \( \Rightarrow \) can set \( s(AO) > 0 \) to put \( A \) in equilibrium \( \Box \)