Flips & friends: (recall from Lecture 5)

Pocket of 2D polygon = region outside polygon & inside convex hull

Pocket lid = convex-hull edge

Flip = reflect pocket through its lid
  = rotate 180° through 3D around the lid
  - avoids self-intersection (line of support)
  - increases area

"Erdős-Nagy" theorem: [posed by Erdős 1935]
  any polygon always convexifies after finite flips, no matter how flip sequence is chosen
  - but can be arbitrarily many:
    [Joss & Shannon 1973]
    - OPEN: bound # flips in n & r = max. dist./min. dist
    - pseudopolynomial? [Overmars 1998]
"Proofs" of Erdős-Nagy Theorem: [Demaine, Gassend, O'Rourke, Toussaint 2007]

- **Nagy 1939**
  - flawed: "$P^0 \subseteq C^0 \subseteq P^1 \subseteq C^1 \subseteq \ldots$"
  - (used to "prove" limit polygon convex)

- **Reshetnyak 1957**
  - correct (though somewhat imprecise)

- **Yusupov 1957**
  - flawed: "limit convex else flip"
  - more subtle error

- **Bing & Kazarinoff 1959**
  - correct (though somewhat terse)

- **Wegner 1993**
  - flawed: "move vertex $\Rightarrow$ increase area by incident $\Delta$"

- **Grünbaum 1995**
  - omission: why limit polygon is convex

- **Toussaint 1999/2005**
  - flawed: "limit convex else flip"

- **Demaine et al. 2007**
  - generalization to self-crossing assuming no "hairpins":
Proof of "Erdős–Nagy" Theorem: \cite{BingKazarinoff1959} & \cite{CCCG2006}

consider an infinite flip sequence

1. distance from a vertex to fixed point \( x \) inside the polygon (remains so) only increases
   - pocket lid is Voronoi diagram of old & new

2. each vertex approaches a unique limit
   - apply 1 to three noncollinear points \( x_1, x_2, x_3 \) inside the polygon
   - distances from vertex \( \leq \) perimeter of polygon/2
   \( \Rightarrow \) distances converge
   \( \Rightarrow \) vertex approaches intersection of 3 circles

3. turn angle at each vertex converges
   - by 2, 3 vertices defining the angle converge
   - by 1, vertices do not get closer to each other
   - rest by continuity

4. vertex moves infinitely \( \Rightarrow \) asymptotically flat
   - each move negates sign of turn angle \( \Rightarrow \) \( 0 \)

5. contradiction
   - eventually asymptotically pointed vxs. stop moving
   \( \Rightarrow \) attain limit convex hull, but about to flip! \( \blacksquare \)
**Flipturn:** rotate pocket 180° in 2D around lid midpoint
- at most \( n! \) configurations \cite{JossShannon1973}
- always \( O(n^2) \) flipturns \cite{Aichholzeretal2002, Ahnetal2000}
- sometimes \( \Omega(n^3) \) flipturns \cite{Biedl2004}
- final polygon & location determined
- NP-hard to find longest flipturn sequence
- **OPEN:** finding shortest flipturn sequence? \cite{Aichholzeretal2002}

**Orthogonal polygons:** \( <n \) flipturns
- count brackets: \[ [ & ] \]
- allow overlap \[ \square \] \( \Rightarrow \leq n \) brackets
- claim # brackets never decreases (13-case analysis)
- orthogonal flipturn kills two brackets:

\[ \Rightarrow \leq n/2 \text{ orthogonal flipturns} \]
- diagonal flipturn kills two vertices:
\[ \Rightarrow < \frac{n}{2} \text{ diagonal flipturns} \]
\[ \Rightarrow < n \text{ total} \]

**OPEN:** \( n-O(1) \) flipturns ever possible?
- best example requires \( \frac{1}{6}n - O(1) \)
Flipturns: (cont'd)

General polygons: ≤ ns if s distinct slopes
- discrete turn angle = 1 + # slopes between
- measure total discrete turn angle:
  - nondegenerate flipturn decreases by ≥ 2
  - degenerate flipturn doesn't change
- also count brackets:
  - nondegenerate flipturn increases by ≤ 2
  - degenerate flipturn decreases by ≥ 2
- potential function = total disc. angle + \( \frac{1}{2} \) # brackets
  - any flipturn decreases by ≥ 1
  - initially ≤ n(s - 1) + n = ns
Algorithms for Alexandrov’s Theorem: (again)
- constructive proof of [Bobenko & Izmestiev 2006] is not technically an algorithm
- need to approximate differential equation by taking “small enough” steps
  - can change $k$’s only slightly & maintain $r$’s
  - triangulation must flip to maintain convexity
- when is step size small enough?
- must also be large enough to guarantee termination

OPEN: finite algorithm based on this construction
OPEN: [Carola Wenk – Nov. 2007] can you compute times at which triang. would flip & set step size = min?

OPEN: (pseudo)polynomial-time algorithm?
OPEN: [Joseph O’Rourke – Nov. 2007] how many flips can there be?
OPEN: [Boris Aronov – Nov. 2007] can a quad flip & later flip back?

also the first step: Delaunay triangulation
  - definitely finite
  - is it (pseudo) polynomial?
Delaunay triangulation algorithm: [Bobenko & Springborn 2006]
- start from some geodesic triangulation
e.g. repeatedly add noncrossing shortest paths
- while some edge not locally Delaunay:
  - flip it

Lemma: if not locally Delaunay then flippable:
  - two distinct triangles (topology)
  - intrinsically convex quad. (geometry)

Lemma: Delaunay flip decreases the sum of areas of \( \Delta \) circumcircles [Tely - PhD 1992]
  (& Musin's harmonic index) (computation)

Lemma: finitely many geodesics from \( p \) to \( q \)
  of length \( \leq L \), for any \( L \geq 0 \)
  (nontrivial)

\[ \Rightarrow \] finitely many triangulations with \( \Sigma \) areas \( \leq A \)
\[ \Rightarrow \] finitely many flips

OPEN: (pseudo)polynomial if initially shortest paths?