Folding polyhedra:
Decision problem:
given a polygon (or connected metric polygonal 2-manifold), can its boundary be glued to itself (in pairs of intervals) such that resulting surface can be folded into exactly a convex polyhedron?

Enumeration problem: list all gluings & foldings
Combinatorial problem: how many can there be?

Why convex polyhedra?
OPEN: if the goal is any nonconvex polyhedron without boundary, is the answer YES for all polygons? [O'Rourke 2004]

Alexandrov gluing: polygon + gluing induce a metric by shortest-path lengths between all pairs of points
- metric is polyhedral: all but finitely many points have zero curvature
- metric is convex if all points have zero or positive curvature
- metric is topological sphere if gluing noncrossing shortest paths from x to all vxs
Alexandrov's Theorem: [1941; English book 2005]
every convex polyhedral metric, topologically a sphere,
is realized by a unique convex polyhedron
(possibly degenerating to doubly covered flat polygon)

Proof sketch:
Uniqueness: draw all shortest paths between pairs of vxs.
- includes all edges of any polyhedral realization
⇒ faces between mesh of paths are rigid
- Cauchy's Rigidity Theorem ⇒ unique convex realiz.

Existence: induct on $n = \# \text{vertices}$
- base case: $n \leq 4$ (double triangle or tetrahedron)
- total curvature of all vertices $= 720^\circ = 4\pi$
  [Descartes' theorem; conseq. of Gauss-Bonnet Formula]
- $n \geq 5$ ⇒ 2 vertices $x, y$ have curvatures $\alpha, \beta < 180^\circ$
- along shortest path from $x$ to $y$,
paste edge of a doubly covered triangle
⇒ new vertex @ triangle apex; adds material @ $x \& y$
- continuously vary angles of triangle at $x \& y$
  from $\emptyset$ to $\alpha/2 \& \beta/2$ ⇒ $x \& y$ flatten
⇒ continuous path on manifold of metrics
  from original metric to metric with one less vertex
- induct on latter
- argue continuity of realizability using
  Implicit Function Theorem ⇒ nonconstructive $\square$
Algorithm for Alexandrov's Theorem: [Bobenko & Izmestiev 2006]
(following Blaschke & Herglotz 1937; Alexandrov 1950; Volkov 1955)

Idea: represent interior of polytope, not just boundary
- add (hypothetical) point p interior to polytope
- triangulate surface with geodesics
- form solid tetrahedron on p & each Δ
- solve for distance \( r_i \) from p to vertex \( v_i \)
⇒ determines geometry of tetrahedra, hence polytope

Generalized polytope: same combinatorial structure, tetrahedra glued around p, but not necc. in 3D
- consider dihedral angles of edges of tetrahedra ~ view as angle of solid material
- convexity invariant: \( \Sigma \) two dihedral angles incident to edge of surface triangulation ≤ 180°
- goal: reach real polytope where \( \chi_i = 360° - \Sigma \) dihedral angles around interior edge \( (p, v_i) = 0 \)

Evolution: start at generalized polyhedron \( P(\emptyset) \)
- set \( \chi_i(t) = (1-t)\chi_i(\emptyset) \to 0 \) as \( t \to 1 \)
- differential equation to evolve \( r_i \)'s:
  \[
  \frac{d\mathbf{r}}{dt} = \left( \frac{\partial\mathbf{P}}{\partial \mathbf{v}} \right)^{-1} \cdot \mathbf{P}(\emptyset)
  \]
  Jacobian - how \( r_i \)'s affect \( \chi_j \)'s
- geodesic triangulation changes (flips) as \( t \to 1 \)
- crucial part of proof: Jacobian has inverse
Algorithm for Alexandrov's Theorem: (cont'd)

Starting point: need generalized polyhedron $P(\emptyset)$

1. Compute Delaunay geodesic triangulation of surface [Bobenko & Springborn 2005]
   - Start with arbitrary geodesic triangulation
   - Flipping algorithm: if circumcircle of edge contains a vertex, flip $e$ $e \Rightarrow$ $e$
   - In 2D, $O(n^3)$ flips suffice
   - Here, can be arbitrarily many ~ but finite
   - Example: can start with "barber pole":
     infinitely many geodesic triangulations!
     cube with triangulated top & bottom; nasty geodesics on side

2. Show that setting all $r_i$ equal & sufficiently large yields desired convexity invariant
   - Using Delaunay property

OPEN: bound on running time?
Ungluable polygon: [Demaine, Demaine, Lubiw, O'Rourke 2000]
- no vertex can be glued into red reflex vertex: $< 90^\circ$ free
  $\Rightarrow$ "zip" red reflex vertex
  $\Rightarrow$ green reflex vertices glued together
  $\Rightarrow > 360^\circ$ of material □

Random polygons are ungluable:
- suppose uniform distribution on angles & edge lengths
  $\Rightarrow \approx \frac{1}{2}$ reflex vertices
- gluing in a convex vertex still leaves reflex vertex
  (angles don't match)
- at some point must zip a reflex vertex
- fails if nearer angle is reflex:

  $\begin{array}{c}
  \text{convex} \\
  \Rightarrow \text{OK}
  \end{array}$

  $\begin{array}{c}
  \text{reflex} \\
  \Rightarrow \text{BAD}
  \end{array}$

- happens with probability $\frac{1}{2}$
  for each reflex vertex □
Perimeter halving: every convex polygon has an Alexandrov gluing
- pick any point \( x \) on polygon boundary
- glue together two boundary points at distance \( d \) from \( x \) (measured along boundary). for all \( d > 0 \)
  - both points have \( \leq 180^\circ \) of material \( \Rightarrow \) convex
- stop at diametrically opposite point \( y \)
\( \Rightarrow \) gluing two halves (paths) of perimeter from \( x \) to \( y \)
- \( x \& y \) also convex (nothing glued)
\( \Rightarrow \) Alexandrov

**EXPERIMENT:** cut out convex polygon
tape together perimeter halves
see what convex polyhedron you get

 Mostly different: uncountably many polyhedra
- vary \( x \) near vertex \( v_i \) say \( d \) along edge \( v_i, v_{i+1} \)
- \( x \) \& \( v_i \) become distinct vertices of shortest-path distance \( d \)
- only finitely many vertex-vertex shortest paths for a particular polyhedron
- uncountably many choices for \( d \)
\( \Rightarrow \) uncountably many polyhedra
Gluing tree:
- turn polygon “inside-out”
- gluing of that boundary to self forms a cycle around a tree
- corresponds to cutting tree in unfolding

Properties:
- each leaf is either a zipped vertex or a fold point in middle of edge (\(\Rightarrow 180^\circ\))
  \(\Rightarrow\) at most 4 fold points (720° total curvature)
- if 4 fold points, then these are only leaves
  \(\Rightarrow\) always induce curvature
- at most one nonvertex (middle of edge) glued at \(\geq 3\)-way junction (else 180° \(+\) something)
*Rolling belt* = path in gluing tree whose end points are either fold pts. or convex vx. leaves & along which always ≤ 180° material on either side = effectively an embedded convex polygon

⇒ can perimeter halve arbitrarily = "rolling the belt"
- only way to get infinite gluings

**Examples:**
1. **Rolling belt:** perimeter halving of convex polygon
2. **Rolling belts:** cylinder
   - belt between every pair of leaves
3. **Rolling belts:**
   - belt between every pair of leaves

≥ 4 **rolling belts**: impossible [6.885 Fall 2004 PS5.3]
- must be 4 fold points
⇒ no curvature elsewhere
⇒ rolling belt from one fold point is uniquely determined to some fold point
⇒ same rolling belt from latter fold point
⇒ ≤ 2 rolling belts