
Every connected union of polygons in 3D, each with a specified visible color (on each side), can be folded from a sufficiently large piece of bicolor paper of any shape (e.g., square).

Proof: fold paper down to long narrow strip (!
- triangulate the polygons
- choose a path visiting each triangle at least once
- cover each triangle along the path by zig-zag parallel to next edge, starting at opposite corner:

Choose parity of zig-zag to arrive at correct corner for next triangle

- turn gadget implements zig-zags & vertex turns:

1. desired turn
2. fold bottom layer
3. hide excess (many folds)
Proof of folding any shape: (cont'd)

- hide excess paper underneath each triangle:
  (more generally, can hide under any convex polygon)
  repeatedly mountain fold along lines extending desired edges

- color-reversal gadget along transition between triangles of opposite colors:
  1. 
  2. 
  3. 
  4. 
  fold bottom layer
Pseudo-efficiency: if allowed to start with any rectangle of paper, then can achieve \( \text{area(paper)} = \text{area(surface)} + \varepsilon \) for any \( \varepsilon > 0 \)

**Proof:** construct Hamiltonian refinement of triangulation:  
- cut each \( \triangle \) into \( \ast \)  
- walk around spanning tree of original dual:  
  - now visit each triangle exactly once  
  - wastage from turns \( \to 0 \) with strip width. \( \square \)

OPEN: pseudopolynomial upper bound? lower bound?

**Seam placement:** can place seams (visible creases/paper boundary) as desired, provided regions between seams are convex  
- idea: vary strip width, use hide gadget

OPEN: what seam placements are possible?
**OPEN:** can a given polygon of paper fold into a given target polygon? Likely NP-hard.

**OPEN:** what is the smallest square that can fold into a given shape? NP-hard?

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**Cube wrapping:** [Catalano-Johnson & Loeb 2001]

- Consider a $1 \times 1$ square.
- In a $x \times x \times x$ cube, every point has an antipodal point at least $2x$ away.
- Center of square must be at least $2x$ away from corner (points only get closer by folding).
- Opposite corners have distance at least $4x$.
- Side length $\geq 2\sqrt{x}$.
- $x \leq \frac{1}{2\sqrt{2}} = \sqrt{1}/4$ & this is possible.

**OPEN:** optimal square $\rightarrow$ regular tetrahedron?

**OPEN:** $x \times y$ rectangle $\rightarrow$ largest cube?  
- Strip method efficient as $x/y \to \infty$.

**OPEN:** optimal square $\rightarrow$ unit $k \times k$ checkerboard  
- Conjecture: $k/2$ for even $k \geq 4$  
- No real lower bounds  
- Seamless?
Tree method: [Lang 1994–2003; Lang & Demaine 2004–] algorithm to find folding of smallest square into “uniaxial” origami base whose projection is a desired metric tree

But: optimization is difficult: exponential time, as hard as disk packing, but good heuristics
- non-self-intersection is only conjectured (we’re working on it)

Uniaxial base:
1. in \( z \geq 0 \) half-space
2. intersection with \( z = 0 \) plane = projection onto that plane
3. partition of faces into flaps, each projecting to a line segment \( \Rightarrow \) all faces vertical
4. hinge crease shared by two flaps projects to a point: common endpoint of flap projections
5. graph of flap projections as edges, connected when flaps share a hinge crease, is a tree (shadow tree). Hinge creases projecting to a vertex form a hinge
6. only one point of paper folds to each leaf
Tree method: (cont’d)

Key lemma: in any uniaxial base from convex paper, distance between two points on shadow tree ≤ distance between corresponding points on paper

Proof: latter = length of line segment on paper
- folds to path in uniaxial
- projects to shorter path on shadow tree
- shortest path in tree is only shorter □

Scale optimization: focus on shadow leaves & placement as points p_i on paper:

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\begin{align*}
\text{maximize} & \quad \lambda \\
\text{subject to} & \quad d(p_i, p_j) \geq \lambda \cdot d(i,j) \quad \text{for leaves } i, j
\end{align*}
\]

\[
\begin{align*}
\text{distance on paper} & \quad \text{fixed distance in tree} \\
\text{quadratic constraint} & \quad
\end{align*}
\]

Example:

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\begin{align*}
\text{star} & \quad \rightarrow \quad \text{disk packing, centers in square} \\
\Rightarrow & \quad \text{with nxn piece of paper, get } (n+1)^3 \\
\Rightarrow & \quad \text{arms in star can flatten to perimeter } \Theta(n^2) \\
\Rightarrow & \quad \text{MARGULIS NAPKIN PROBLEM} \quad [\text{Lang 2003}]
\end{align*}
\]
Tree method (cont'd)

Active path = path between two shadow leaves
  of length = distance in piece of paper
  - never cross each other  [GFALOP Lem.16.4.2]

Triangulation: can add artificial leaf edges to the
  shadow tree to make the active paths
  partition the piece of paper into triangles
  (without changing scale factor)  [GFALOP, Lem.16.6.2]
  - later these leaf edges can be "folded away"
  - some triangle edges are paper boundary, not active

Rabbit-ear molecule:
  active paths
  angular bisectors
  leaves
  HINGES = perpendiculare emanating
  from tree vertices along path
  - put them together to form entire shadow tree

Example:
More practically:
- use convex decomposition instead of triangulation
  (in practice by letting tree edge lengths vary a bit)
- Lang Universal Molecule folds convex polygon

2 kinds of events:
1) gusset: new active path $\rightarrow$
  split shrunken polygon
2) two vertices meeting $\rightarrow$
  continue along new angular bisector