Graph = vertices & edges
(connectivity/combinatorial structure)

Linkage = graph + lengths of edges (l:E→R≥0)
(intrinsic geometry)
[ + coordinates for pinned vertices (p:V\to R^d) ]

Configuration of a linkage into R^d
= coordinates for vertices (C:V→R^d)
satisfying constraints of linkage
\|C(v)−C(w)\| = l(v,w) for all \{v,w\}∈E ;
C(v) = p(v) for all v∈\text{dom }p
(allowing intersections for this lecture)

Example:

\begin{align*}
\text{graph} & \quad \text{linkage} & \quad \text{two configurations} \\
(0,1) & \quad 1 & \quad 0 \quad \text{circle} & \quad 1 \\
(1,0) & \quad 1 & \quad 0 \quad \text{square} & \quad 1
\end{align*}

Motion (of a linkage in R^d)
= continuum of configurations (m:[0,1]→C)
Configuration space = all configurations of a linkage
- view configuration of n-vertex linkage in \( \mathbb{R}^d \)
  as (special) point in \( \mathbb{R}^{dn} \):
  \[
  C = (\ldots, \underbrace{\ldots, \ldots, \ldots, \ldots}_{d \text{ coords for } v_1}, \underbrace{\ldots, \ldots, \ldots, \ldots}_{d \text{ coords for } v_2}, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)
  \]
  \[d \text{ coords for } v_1 \quad v_2 \quad d \text{ coords for } v_n\]
  \[\Rightarrow \text{configuration space} = \text{subspace of } \mathbb{R}^{dn}\]
- motion = path/curve in configuration space
- square example: \( n=4, d=2 \)
  \[\Rightarrow \text{configuration space lives in } \mathbb{R}^8\]
  - 4 dimensions fixed by pinning
  - locally one dimensional; topologically:

 Degrees of freedom = local intrinsic dimension
  of configuration space around configuration
  - intuitively: \( d \cdot (\# \text{ unpinned vertices}) - (\# \text{ edges}) \)
  (but in reality, some edges are extraneous - see L3)

 Trajectory of a vertex in a linkage
  = all points that vertex can reach in configurations
  (= projection of configuration space onto vertex's coords)
Kempe’s Universality Theorem: [Kempe 1876 had bug; Thurston, King 1999; Kapovich & Millson 2002; Abbott, Barton, Demaine, O’Rourke]

Any algebraic planar curve \( \Phi(x,y) = \sum_i c_i x^{p_i} y^{q_i} = 0 \), intersected with any bounded disk, (necessary) is exactly the trajectory of a vertex of some linkage.

Kempe’s “proof”:
- start with rhombus to constrain point \( p \) within disk:

\[
\begin{align*}
(0,0) & \quad \text{(center of disk)} \\
\text{disk (radius } r) & \\
\end{align*}
\]

- goal: constrain \( p = (x, y) \) to satisfy \( \Phi(x,y) = 0 \)

Main trick: use trig. to effectively “take logarithm”
- \( x = \frac{r}{2} \cos \alpha + \frac{r}{2} \cos \beta \)
- \( y = \frac{r}{2} \sin \alpha + \frac{r}{2} \sin \beta = \frac{r}{2} \cos (\alpha - \frac{\pi}{2}) + \frac{r}{2} \cos (\beta + \frac{\pi}{2}) \)
- apply trig. identity
  \[
  \cos A \cos B = \frac{1}{2} \left[ \cos (A+B) + \cos (A-B) \right]
  \]
  to polynomial \( \Phi(x,y) = \sum_i c_i x^{p_i} y^{q_i} \)
  \[
  \Rightarrow \Phi(x,y) = c + \sum_i c_i \cos^i \left( r_i \alpha + s_i \beta + \delta_i \right)
  \]
  with \( c \) const., \( i \) const., \( r_i \) int., \( s_i \) int., \( \delta_i \) 0 or \( \pm \frac{\pi}{2} \)
Kempe's "proof": (cont'd)

- **new goal:** construct line segment of length $c_i$ & angle $r_i \alpha + s_i \beta + \delta_i$, for each $i$

- force final vertex on line $x = -c$
  - via large Peaucellier linkage

- build "machine" for angle arithmetic with ops:
  - multiply given angle by integer
  - add two given angles
  - copy an angle from one place to another
Kempe's gadgets:

Contraparallelogram:
- opposite sides equal & self-crossing (not parallelogram)
\[ \Rightarrow \] opposite angles equal; \( \alpha \) determines \( \beta \)

Multiplier:
- \( k \) similar contraparallelograms sharing their \( \beta \)'s \( \Rightarrow \) equal \( \alpha \)'s
- can be more efficient — \( O(lg \, k) \) edges — by repeated doubling, but this will not affect final complexity

Additor:
- use \( 2 \times \) multipliers to
  - bisect angle between segments
  - reflect x axis through bisector

Translator: two parallelograms
- opposite edges parallel & same length
- make adjacent edges long (\& same) for reach
- could use big rhombous — but this construction allows arbitrary length of input (or output) edge
Bug: [Kapovich & Millson]
- parallelograms can flip to contraparallelograms & vice versa via degenerate (flat) configuration
  ⇒ Kempe proved weaker result:
  trajectory includes desired poly curve & more
- fix for parallelogram:

- different, messier construction for complex polynomials
- fix for contraparallelogram: [Abbott & Barton 2004]

Generalizations/strengthenings:
- curves/surfaces in d dimensions
- $\Theta(n^d)$ bars is optimal for degree $n$
- any compact semialgebraic set ($d$-dim.)
  (bounded system of polynomial $\leq$ inequalities)
  as vertex trajectory
- configuration space = union of finitely many
  analytically isomorphic copies of any
  desired algebraic set ($any \neq dim.$)
  mapping & inverse have local power-series expansion
- sign name via Weierstrass approximation theorem:
  any continuous function $f: [a, b] \rightarrow \mathbb{R}$ has an $\varepsilon$-approximate
  polynomial $p - |p(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$ for any $\varepsilon > 0$
  (apply to each coordinate of curve)
OPEN: can you actually reach the whole polynomial curve/semi-algebraic set with one continuous turning of crank? (continuity & ideally no branching points)

OPEN: what if edges are forbidden from crossing? [Shimamoto 2004]

PROJECT: implement Kempe applet

PROJECT: sculpture based on Kempe linkage/gadgets

PROJECT: design linkages for letters of alphabet