Rigid Origami Summary

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Things to remember

- By **rigid origami** we mean origami that can be folded while keeping all regions of the paper flat and all crease lines straight.
- We try to study rigid origami by making a **mathematical model** of it.
  There is more than one way to do this:
  - a matrix model
  - a geometric model (Gaussian curvature)

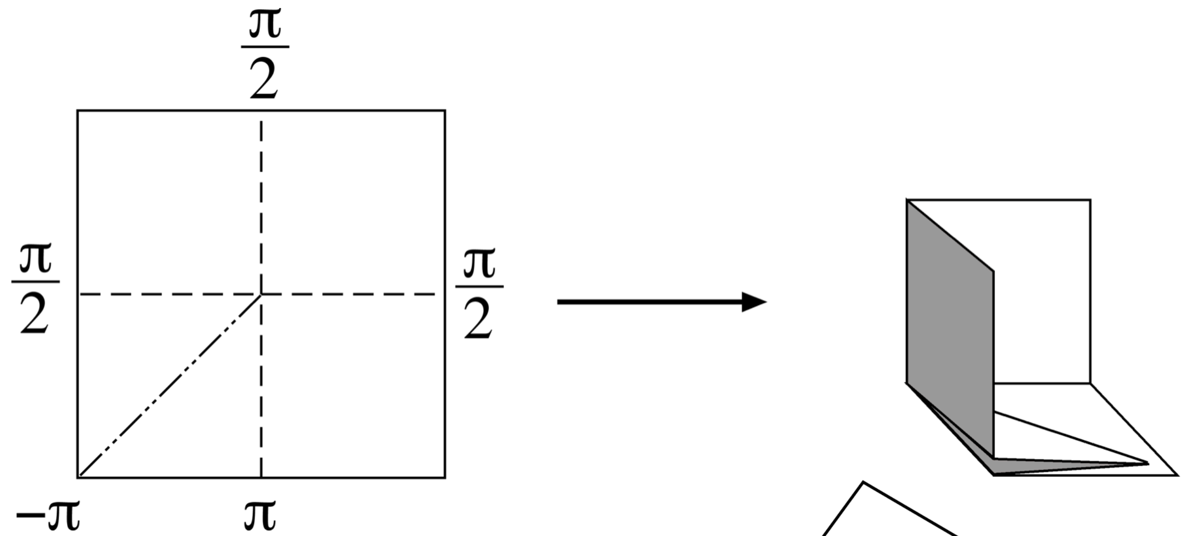
We **hope** that such models will be able to answer our questions about rigid folds.

What are the questions?

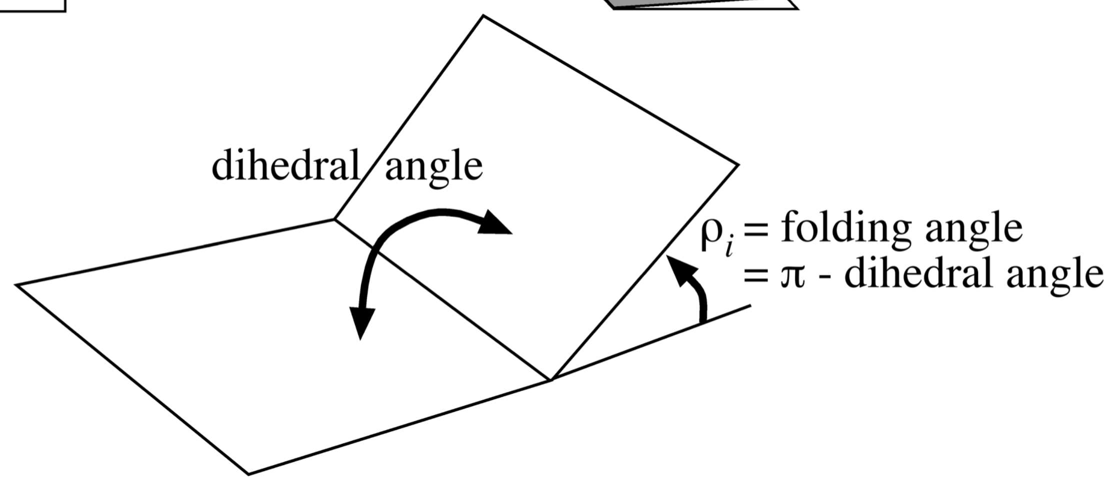
- Given an origami fold, how can we prove that it is rigid / not rigid?
- Can rigidity analysis tell us anything about how such origami folds and unfolds continuously?

Matrix Model

The idea: rigid motions of the plane are isometries, and so can be modeled with linear transformations.



Note: the angles shown above are the folding angles.



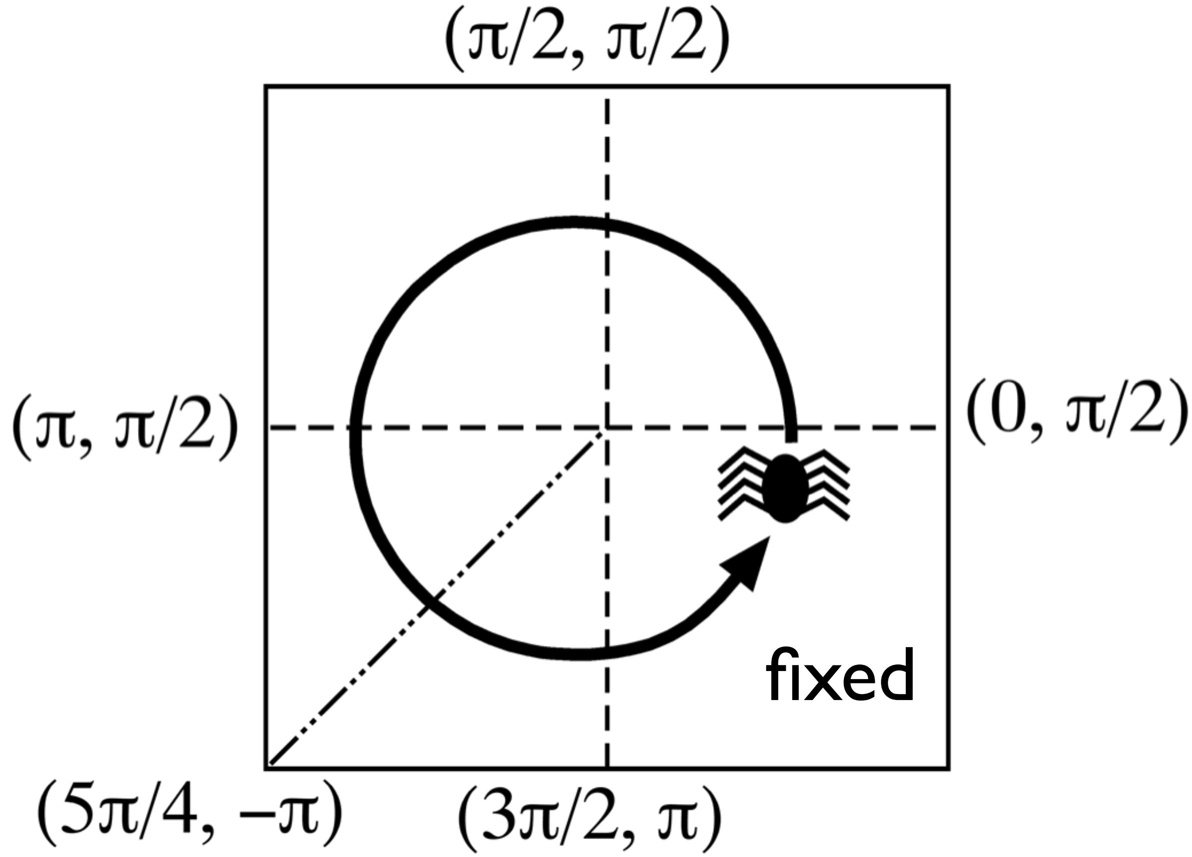
Matrix Model

Modeling a single rigid vertex: imagine a spider walking around the vertex on the folded paper.

Here each crease is labeled with (α_i, ρ_i), where

α_i = position angle

ρ_i = folding angle



Matrix Model

Let χ_i = rotation counterclockwise by angle ρ_i about crease line l_i (which lies in the xy plane).

First crease the spider crosses: rotation is $L_1 = \chi_1$

Second crease: rotation is $L_2 = L_1 \chi_2 L_1^{-1} = \chi_1 \chi_2 \chi_1^{-1}$

Third crease: $L_3 = L_2 L_1 \chi_3 L_1^{-1} L_2^{-1} = \chi_1 \chi_2 \chi_3 \chi_2^{-1} \chi_1^{-1}$

ith crease: (redo previous Ls)• χ_i • (undo previous Ls) in reverse order

$$L_i = (L_{i-1} \ldots L_1) \chi_i (L_1^{-1} \ldots L_{i-1}^{-1}) = \chi_1 \cdots \chi_{i-1} \chi_i \chi_{i-1}^{-1} \cdots \chi_1^{-1}$$

Matrix Model

Necessary condition for rigidity: the spider must come back to where it started.

In other words, $L_n L_{n-1} \ldots L_1 = I$.

Since $L_i = \chi_1 \ldots \chi_{i-1} \chi_i \chi_{i-1}^{-1} \ldots \chi_1^{-1}$ this simplifies to:

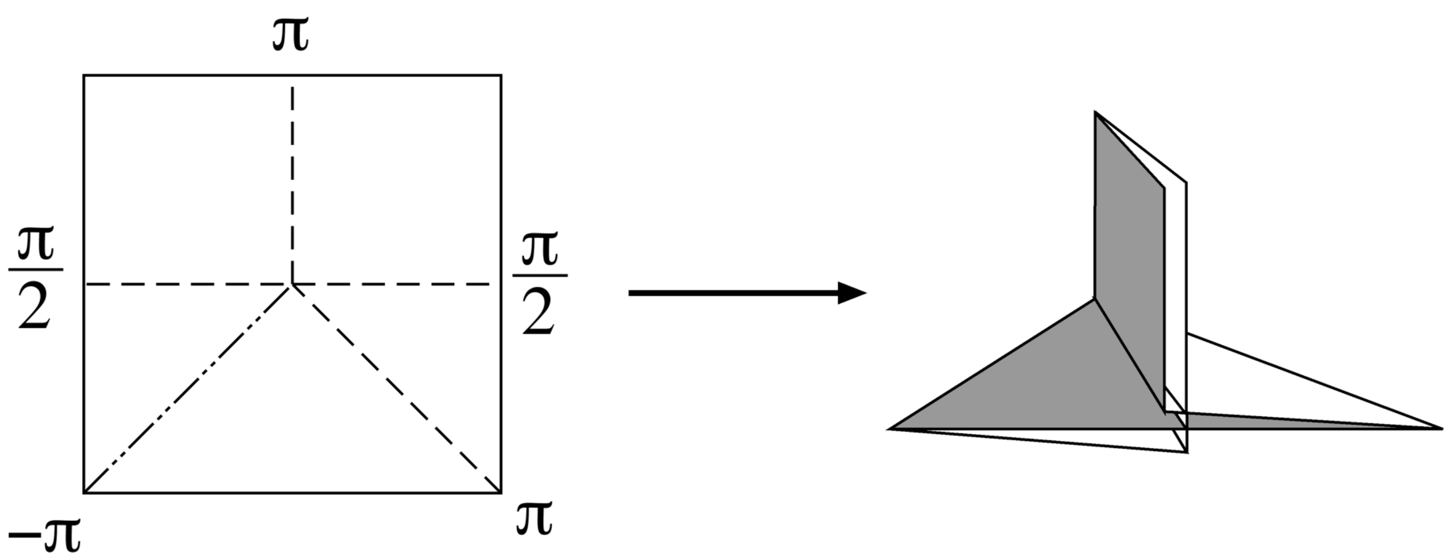
$$\chi_1 \chi_2 \cdots \chi_n = I$$

(credits, Kawasaki 1996, belcastro and Hull 2003)

Matrix Model

Problems with this model:

- It's not a sufficient condition.
  Example:



This satisfies $\chi_1 \chi_2 \ldots \chi_n = I$ but it self-intersects.

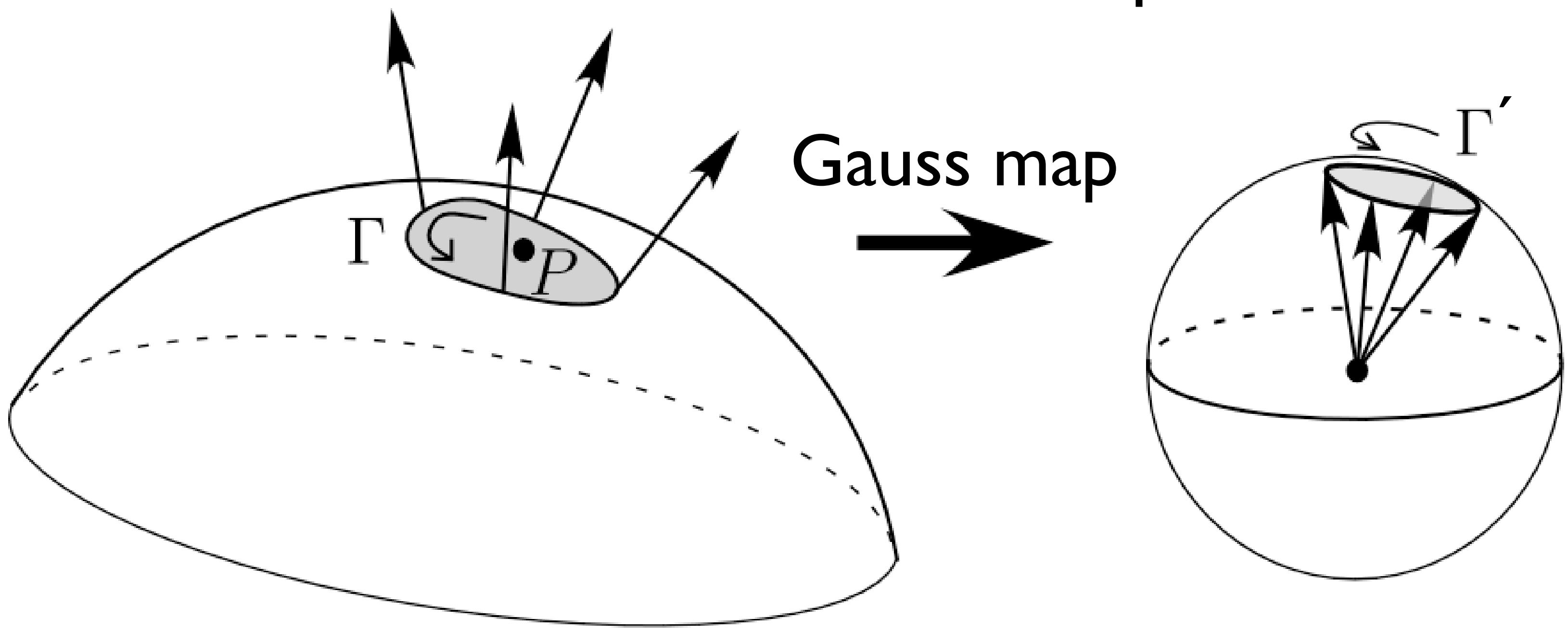
Matrix Model

Problems with this model:

- It's not a sufficient condition.
- It does not say anything about the continuous folding/unfolding process. (That is, can we get from the unfolded state to the desired folded state via a rigid folding motion?)

Gaussian Curvature Model

Definition of Gaussian curvature at a point on a surface:

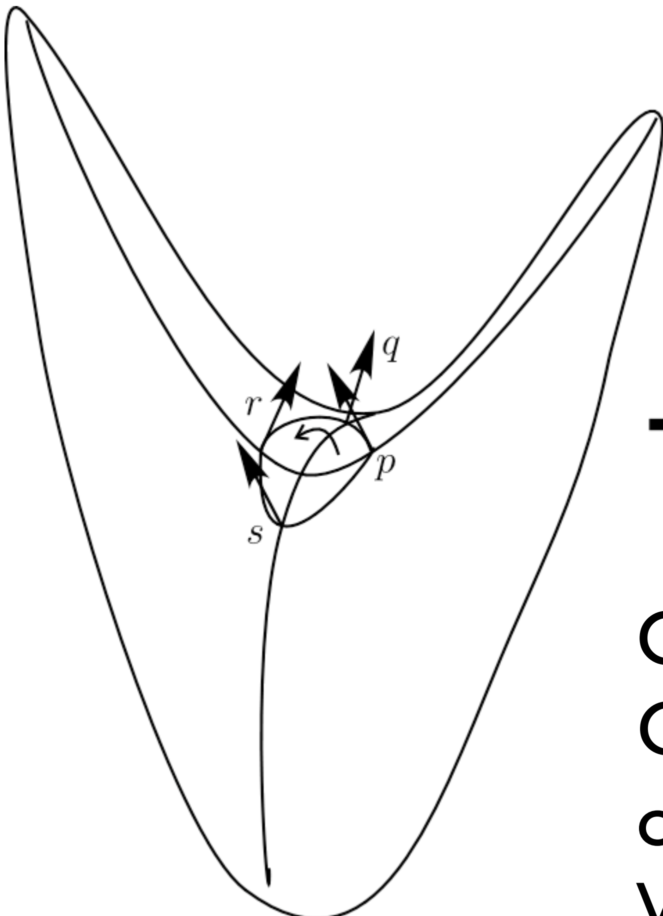


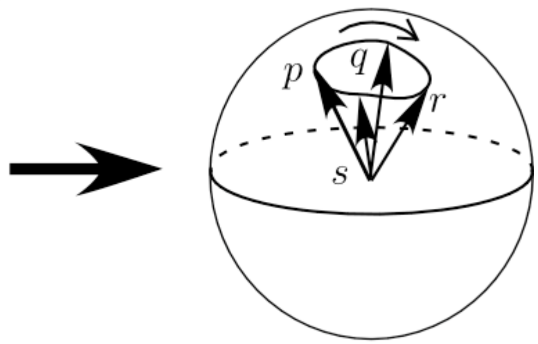
Then the curvature κ of the surface at P is

$$\kappa = \lim_{\Gamma \to P} \frac{\text{Area in } \Gamma'}{\text{Area in } \Gamma}$$

Gaussian Curvature Model

Definition of Gaussian curvature at a point on a surface:

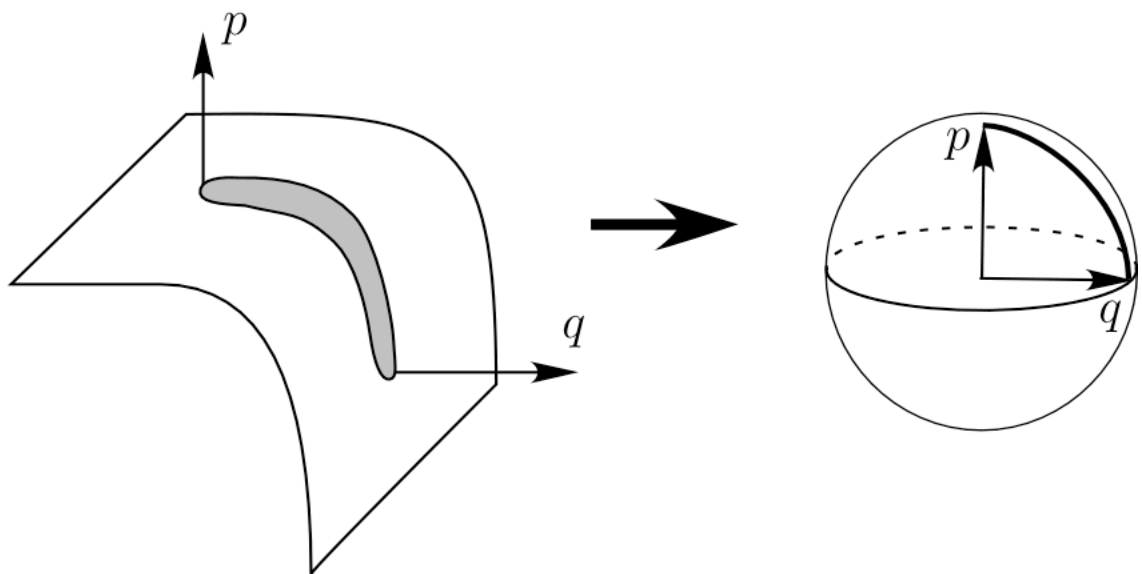




On a "Pringle chip" surface, the Gauss map curve will go in the opposite direction as the original. We consider the area inside such a Gauss map curve to be negative, and we call this **negative curvature**.

Gaussian Curvature Model

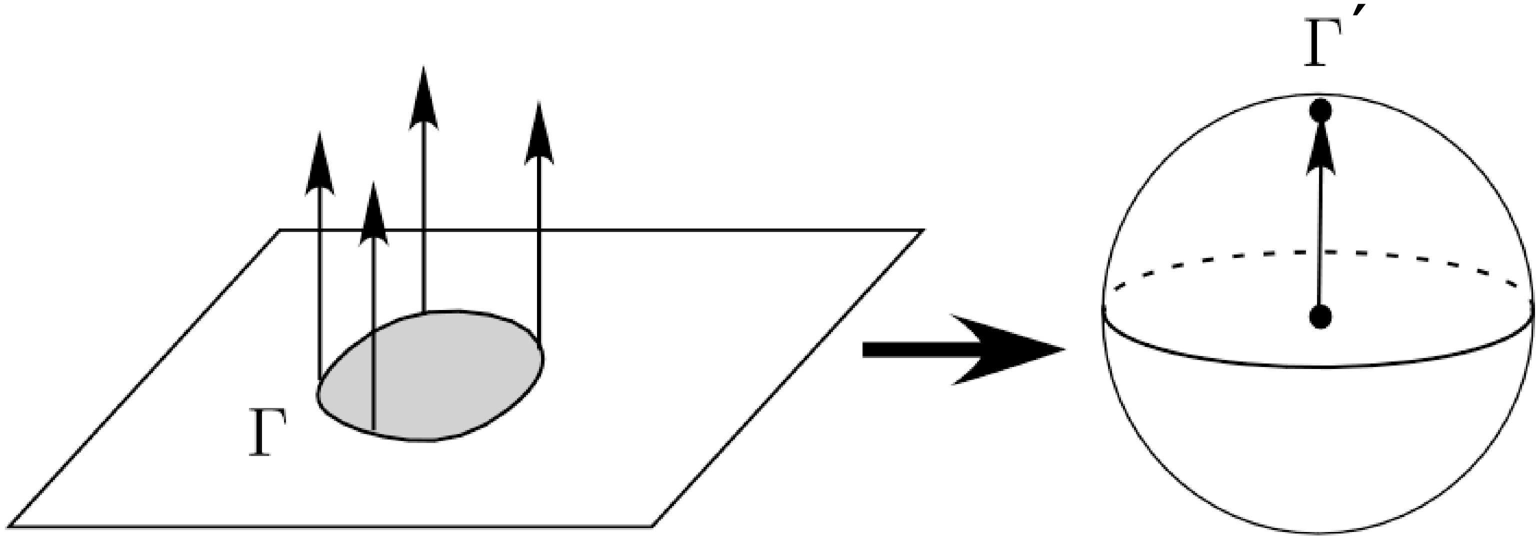
Definition of Gaussian curvature at a point on a surface:



Also, the curvature of a surface should not change if we deform the surface nicely (i.e., bending).

Gaussian Curvature Model

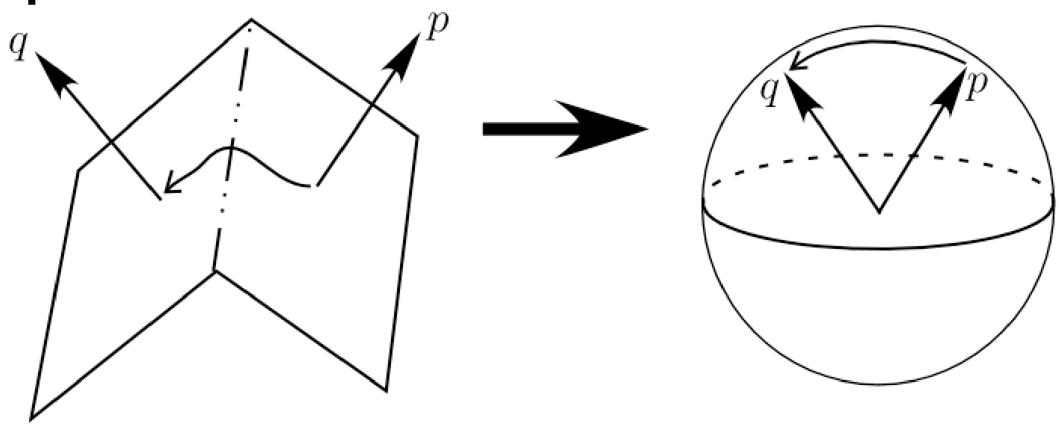
How this applies to origami:



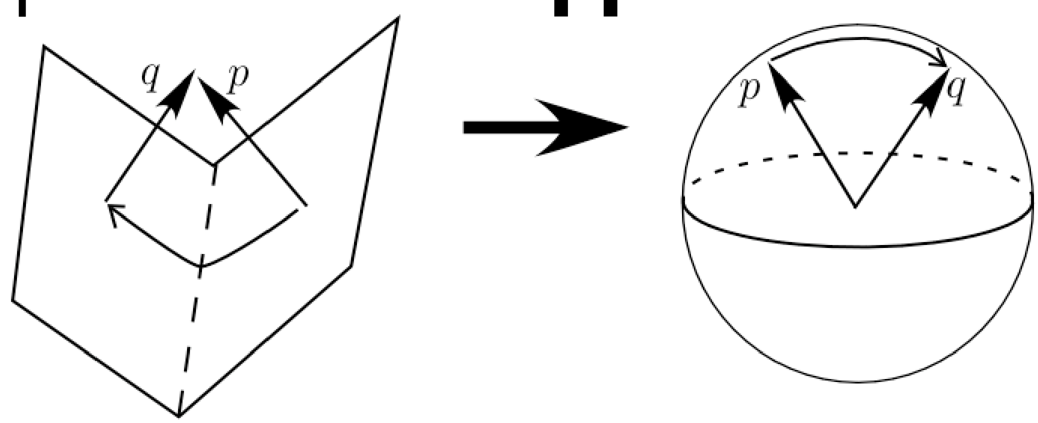
Paper has zero curvature everywhere.

Gaussian Curvature Model

When the curve Γ crosses a mountain crease, the Gauss map travels in the **same** direction.

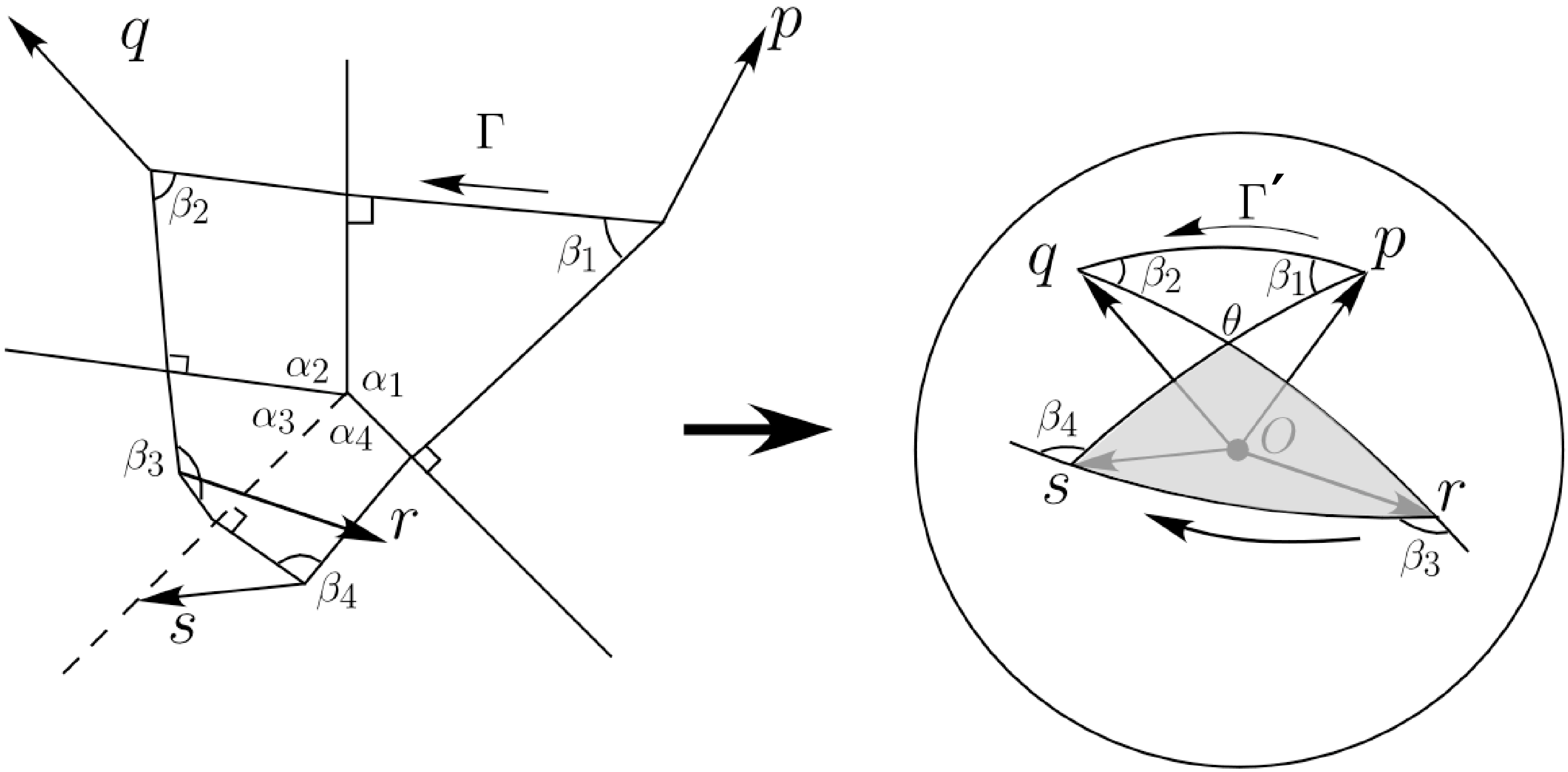


When the curve Γ crosses a valley crease, the Gauss map travels in the **opposite** direction.



Gaussian Curvature Model

Example: a 4-valent vertex fold



This should still have zero curvature! Does it?

Gaussian Curvature Model

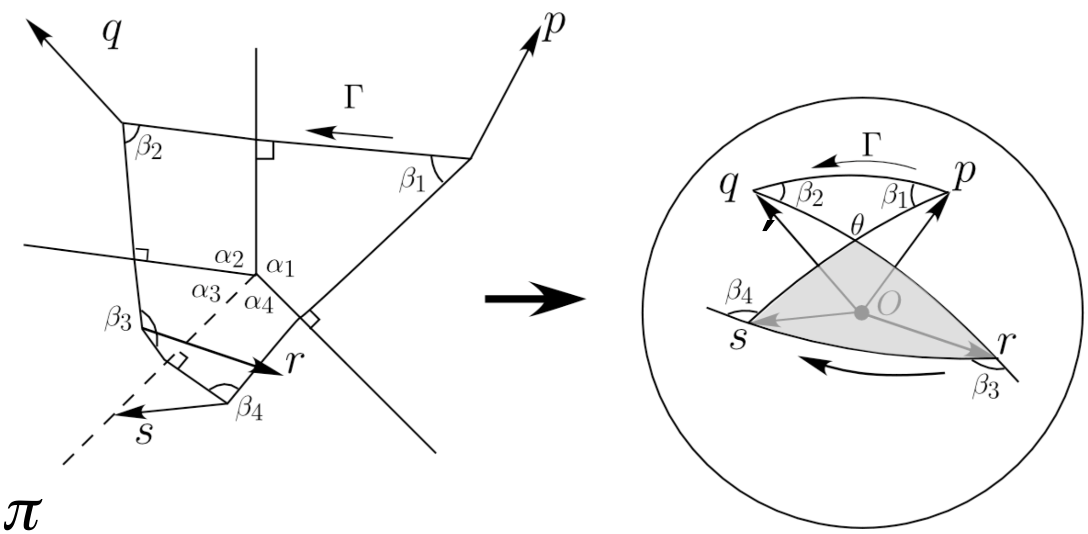
Example: a 4-valent vertex fold

Area of the top triangle=

$$\beta_1 + \beta_2 + \theta - \pi$$

Area of the bottom triangle=

$$(\pi - \beta_3) + (\pi - \beta_4) + \theta - \pi$$



Also notice that $\alpha_i = \pi - \beta_i$

So (Area of Top)–(Area of Bottom)=

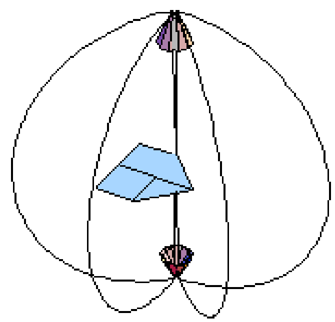
$$((\pi - \alpha_1) + (\pi - \alpha_2)) - (\alpha_3 + \alpha_4)$$

$$= 2\pi - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0.$$

Credits:
Huffman 1976,
Miura 1980.

Gaussian Curvature Model

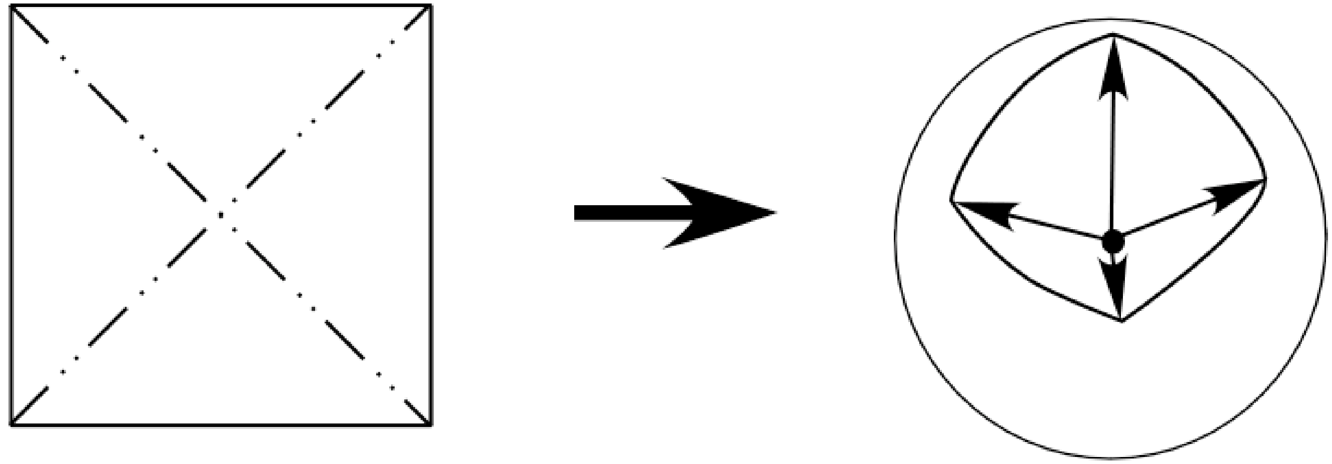
Animations



Gaussian Curvature Model

Proving that some folds are non-rigid:

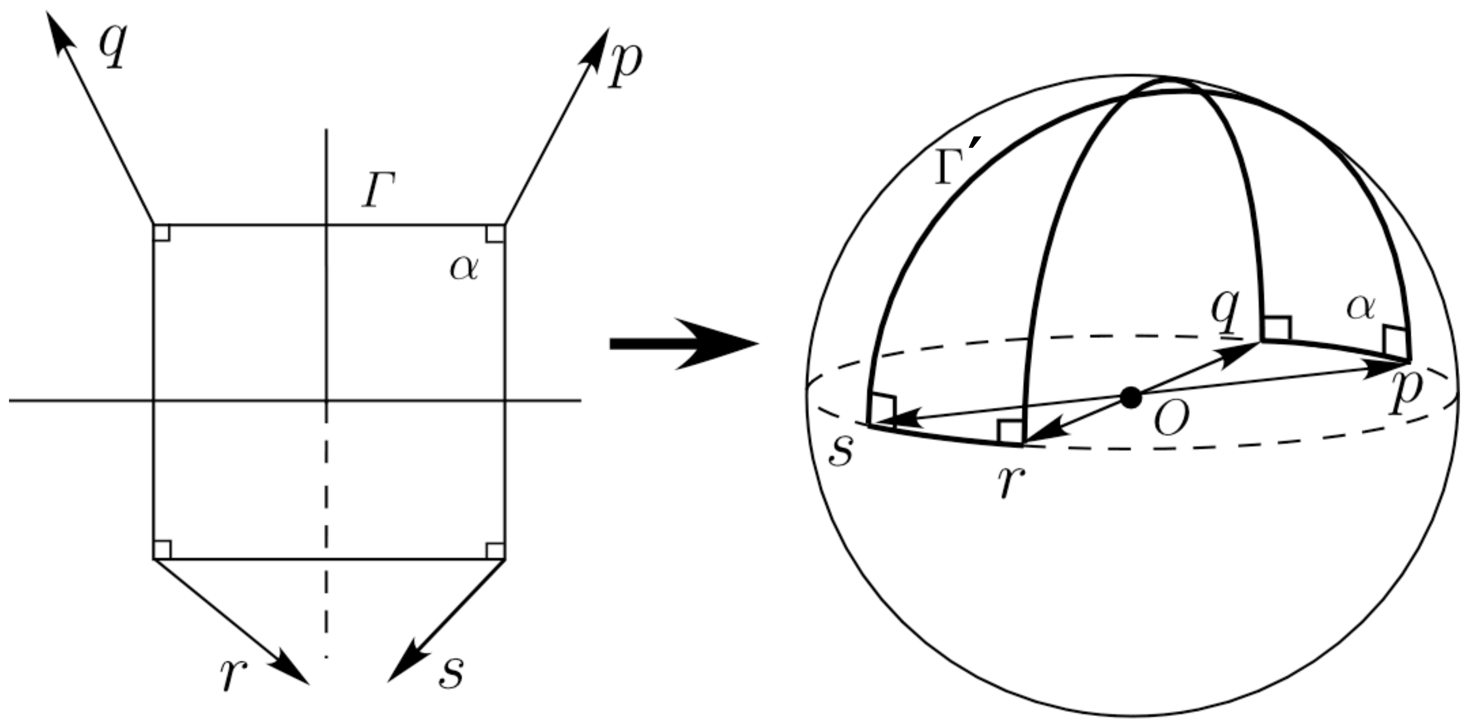
Can a 4-valent, all-mountains vertex be folded rigidly?



Gaussian Curvature Model

Proving that some folds are non-rigid:

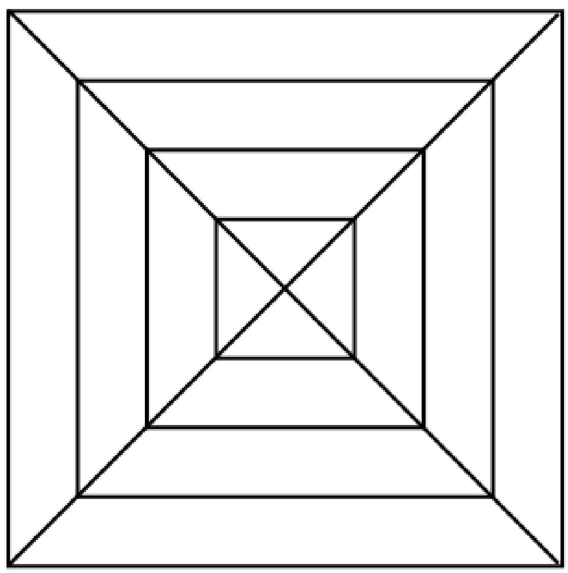
What about 3 mountains, 1 valley, with 90° angles between them?

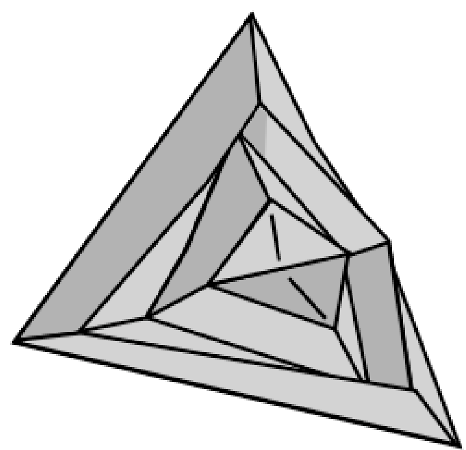


Gaussian Curvature Model

Proving that some folds are non-rigid:

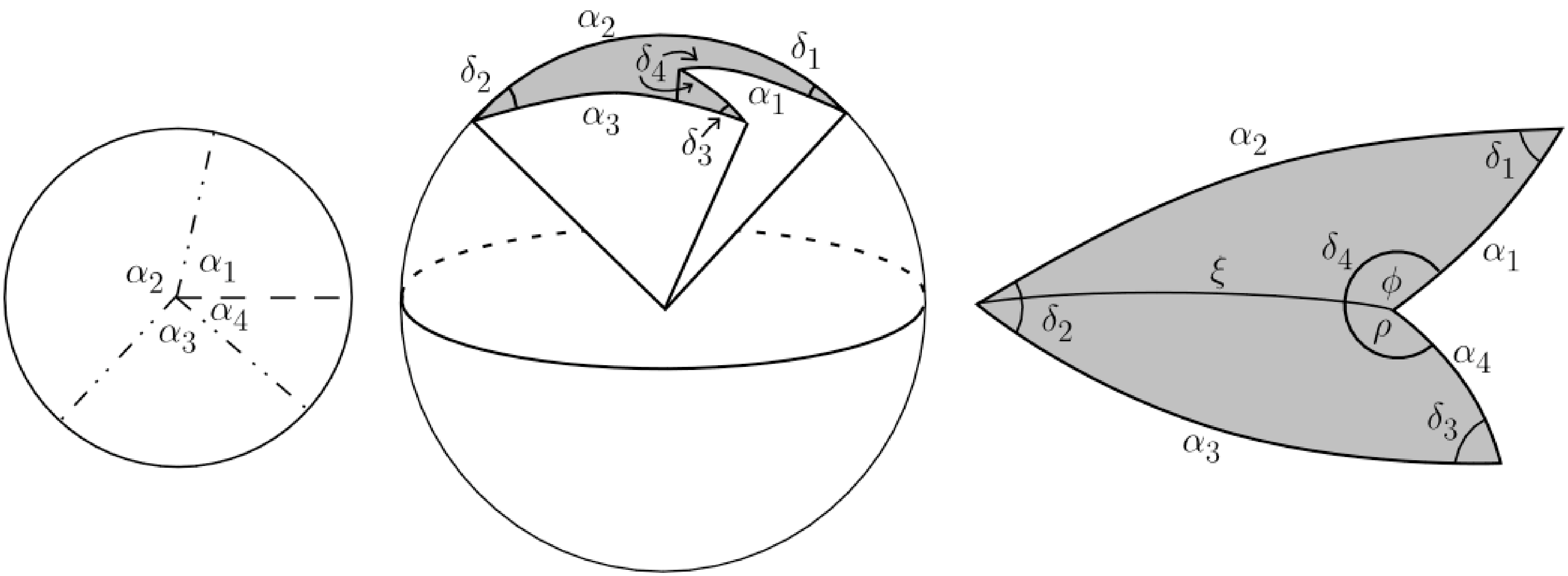
What about the hyperbolic paraboloid?





Focus on 4-valent Flat Vertices

Consider such a partially-folded vertex centered at a sphere of radius one. Then the paper cuts out a spherical (non-convex) quadrilateral on the sphere.



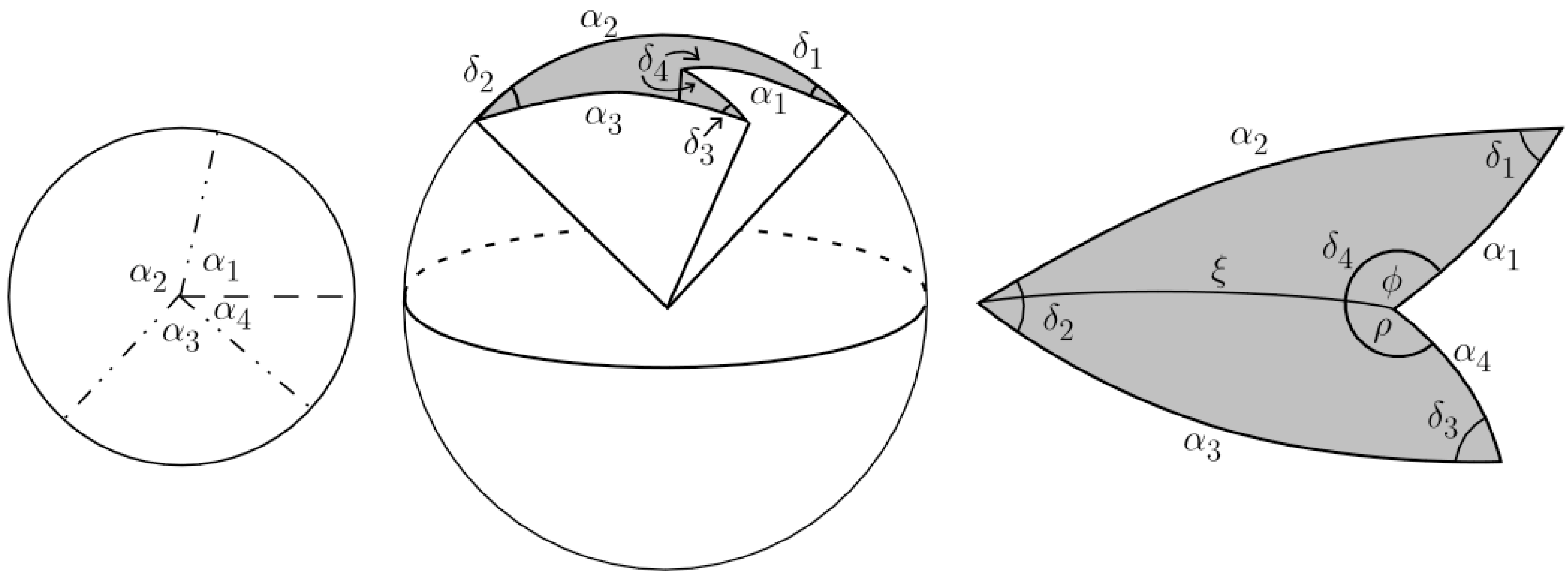
δ_i = the dihedral angle at the ith crease.

Focus on 4-valent Flat Vertices

The spherical law of cosines says:

$$\cos\xi = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2 \cos\delta_1 \quad (1)$$

$$\cos\xi = \cos\alpha_3 \cos\alpha_4 + \sin\alpha_3 \sin\alpha_4 \cos\delta_3 \quad (2)$$



Kawasaki's Theorem says that $\alpha_3 = \pi - \alpha_1$ and $\alpha_4 = \pi - \alpha_2$.

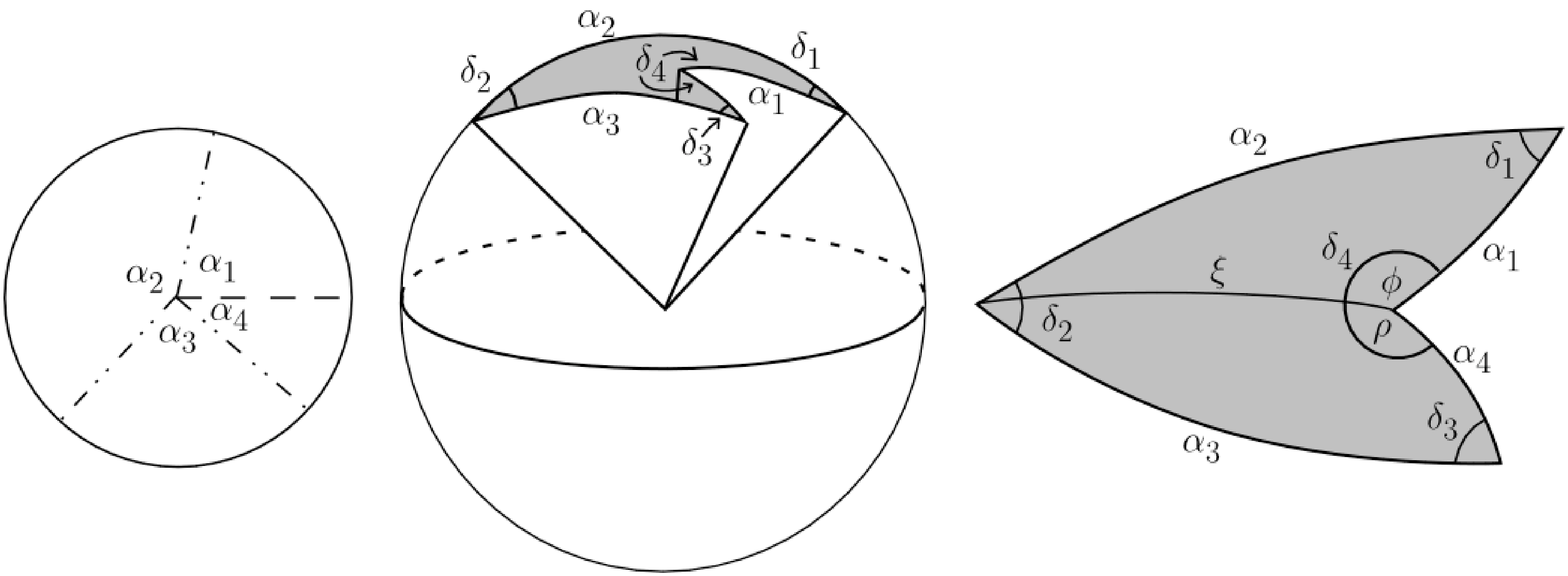
Plugging this into (2) gives $\cos\xi = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2 \cos\delta_3$

Subtract this from (1) and get...

Focus on 4-valent Flat Vertices

$$\sin\alpha_1 \sin\alpha_2 (\cos\delta_1 - \cos\delta_3) = 0$$

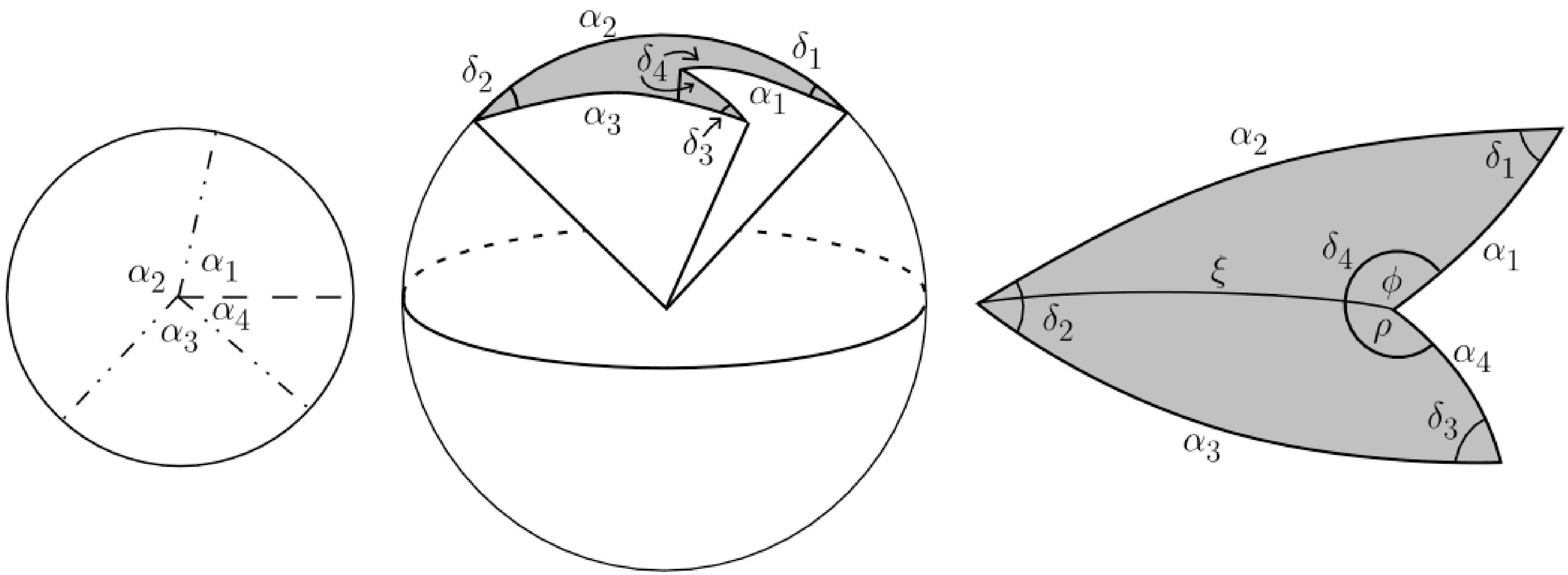
Now, $0 < \alpha_i < \pi$, so those sines can't be zero.



So $\cos\delta_1 = \cos\delta_3$ and $\cos\delta_2 = \cos\delta_4$.
Thus $\delta_1 = \delta_3$ and $\delta_2 = 2\pi - \delta_4$.

Focus on 4-valent Flat Vertices

Now, $\delta_4 = \phi + \rho$, and after some truly yucky spherical trig we can get $\cos\delta_2 = \cos\delta_1 - \dfrac{\sin^2\delta_1 \sin\alpha_1 \sin\alpha_2}{1 - \cos\xi}$



Since cosine is an decreasing function from 0 to π/2, we have $\cos\delta_2 < \cos\delta_1 \Rightarrow \delta_1 > \delta_2$

Focus on 4-valent Flat Vertices

Thus for a 4-valent flat, rigid vertex, we have

- opposite pairs of dihedral angles are "equal."
- the same-parity pair are greater than the other pair.

Corollaries:

One dihedral angle will determine all the others.

The classic square twist is not a rigid fold.

Square Twists