**Exam 2**
Assigned: 05/03/2005
Due: 05/05/2005 (Midnight)

**Requirements:** *The exam is open-book and open-web. (Sources should be cited if you referred to websites or other books.) Take-home exams may not be discussed. No collaboration is allowed. You should include the following sentence in your exam write-up, which you need to sign physically or electronically:* `"I affirm that I did not discuss this exam or receive help from anyone else on it, and that I have disclosed the resources that I used in working on each problem. Signature: ___________ Date: ___________"`
**Submission:** *Submission deadline is 05/05/2005 Midnight. No extension will be granted. Submission can be an electronic version to 6869-submit@csail.mit.edu, or a hard-copy sent to 32D542.*

**Problem 1** *Hough Transform*

Hough Transform can be generalized to detect any curve that can be parameterized. Consider detecting circles in images. To do this, a Canny edge detector can be applied to an image to obtain a set of edge points. Now:

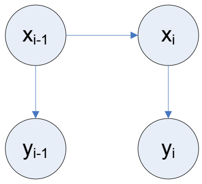
(a) Write down a parametric form of circles. Give a brief description of how to use Hough Transform to detect circles. How many dimensions are needed in the parameter space for this problem?

(b) Now assume that the gradient direction at each edge point is also observed. How does the voting process change? Does the dimensionality of the parameter space change? Why or why not?

(c) Under what conditions can you interpret the table entries of the Hough Transform as log-likelihood values?

**Problem 2** *Kalman Filter as Dynamic Bayesian Network*

In lecture 13, we derived Belief Propagation by marginalizing over hidden variables in the Bayes Network. The graphical model of one step of Kalman filter is shown in the following figure:




(a) Write $P(x_{i-1}, x_i, y_{i-1}, y_i)$ for this network in terms of conditional probabilities between pairs of variables, and the prior probability $P(x_{i-1}|y_{i-2}, y_{i-3}, ...)$ (The prior probability for $x_{i-1}$ given all observations before time $i-1$.).

(b) By marginalizing over $x_{i-1}$, derive the Belief Propagation update equations for this network and show that it will be equivalent to the Kalman filter described in Algorithm 17.1 in section 17.3.1 of Forsythe and Ponce.

Write the formulas for the messages passed between each of the variables, and for the marginal probability at $x_i$ in terms of the messages.

**Problem 3** *Multi-class Clustering*

One way to do multi-class clustering is given in Algorithm 14.6 in Forsythe and Ponce, using graph eigenvectors. Another strategy could be to run the algorithm to separate all samples into two clusters, construct affinity matrix for each new cluster and run the algorithm recursively to divide new clusters until the required number of clusters is reached.
Explain why these two algorithms are not equivalent.

**Problem 4** *Object Recognition using Eigenspaces*

For the eigenface algorithm described in page 510 of Forsythe and Ponce, answer the following true/false questions. For each question, give a brief explanation of your answer.

(a) True or False: The eigenface algorithm is invariant under translation of target objects in test images. That is, it can recognize objects in test images that are translated relative to their position in training images.

(b) True or False: The eigenface algorithm is invariant to illumination intensity change, for example, adding a constant value to each pixel.

(c) True or False: The eigenface algorithm is invariant to illumination direction change.

**Problem 5** *Bayesian Classifier*

Solve exercise 22.2 in Forsythe and Ponce. (Note there is an insignificant typo in the algorithm 22.2. The $\frac{1}{2}$ in the expression for $\delta$ should be inside the square root.)

**Problem 6** *Graphical Model*

Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ be a 4-dimensional Gaussian random variable with zero mean and covariance matrix $\Sigma$ given by:

$$\Sigma = \begin{pmatrix} E(x_1x_1) & E(x_1x_2) & E(x_1x_3) & E(x_1x_4) \\ E(x_2x_1) & E(x_2x_2) & E(x_2x_3) & E(x_2x_4) \\ E(x_3x_1) & E(x_3x_2) & E(x_3x_3) & E(x_3x_4) \\ E(x_4x_1) & E(x_4x_2) & E(x_4x_3) & E(x_4x_4) \end{pmatrix} = \frac{1}{45} \begin{pmatrix} 21 & -9 & 6 & -9 \\ -9 & 21 & -9 & 6 \\ 6 & -9 & 21 & -9 \\ -9 & 6 & -9 & 21 \end{pmatrix}$$

Which one(ones) of the following assertions is(are) true? Why?

(a) $x_1$ is independent of $x_3$

(b) $x_1$ is independent of $x_3$ given $x_2$

(c) $x_1$ is independent of $x_3$ given $x_2$ and $x_4$

**Problem 7** *Graphical Model*

(a) Consider the following model with input variable $x$, output (or target) variable $t$ and hidden (or unobserved) variable $y$, and three parameters $\theta_1$ $\theta_2$ $\theta_3$. $p(t, y, x) = p_{\theta_1}(t|y)p_{\theta_2}(y|x)p_{\theta_3}(x)$. $\theta_1$ is the parameter controlling $p_{\theta_1}(t|y)$, and so on. Use Bayes rule to compute the distribution, $p(y|x, t)$, where the final expression must be in terms of the model distributions $p_{\theta_1}(t|y)$ $p_{\theta_2}(y|x)$ $p_{\theta_3}(x)$ only.

(b) Draw the graphical model and check whether $x$ and $t$ are independent, given $y$. Check this by explicit calculation, i.e. show that $p(t|y, x) = p(t|y)$.

(c) Now add priors for the parameters: $p(\theta_1)$ $p(\theta_2)$ $p(\theta_3)$. The Bayesian expression for the probability distribution becomes, $p(t, y, x, \theta_1, \theta_2, \theta_3) = p(t|y, \theta_1)p(y|x, \theta_2)p(x|\theta_3)p(\theta_1)p(\theta_2)p(\theta_3)$. Draw the corresponding graphical model. Suppose we observed $x$ and $t$, so we shade these nodes. Now check if the following statements are true,

(i) $\theta_1$ is independent of $\theta_2$ given $t$ and $x$.

(ii) $\theta_1$ is independent of $\theta_3$ given $t$ and $x$.