

# 6.867 Machine learning

Mid-term exam

October 18, 2006

(2 points) Your name and MIT ID:

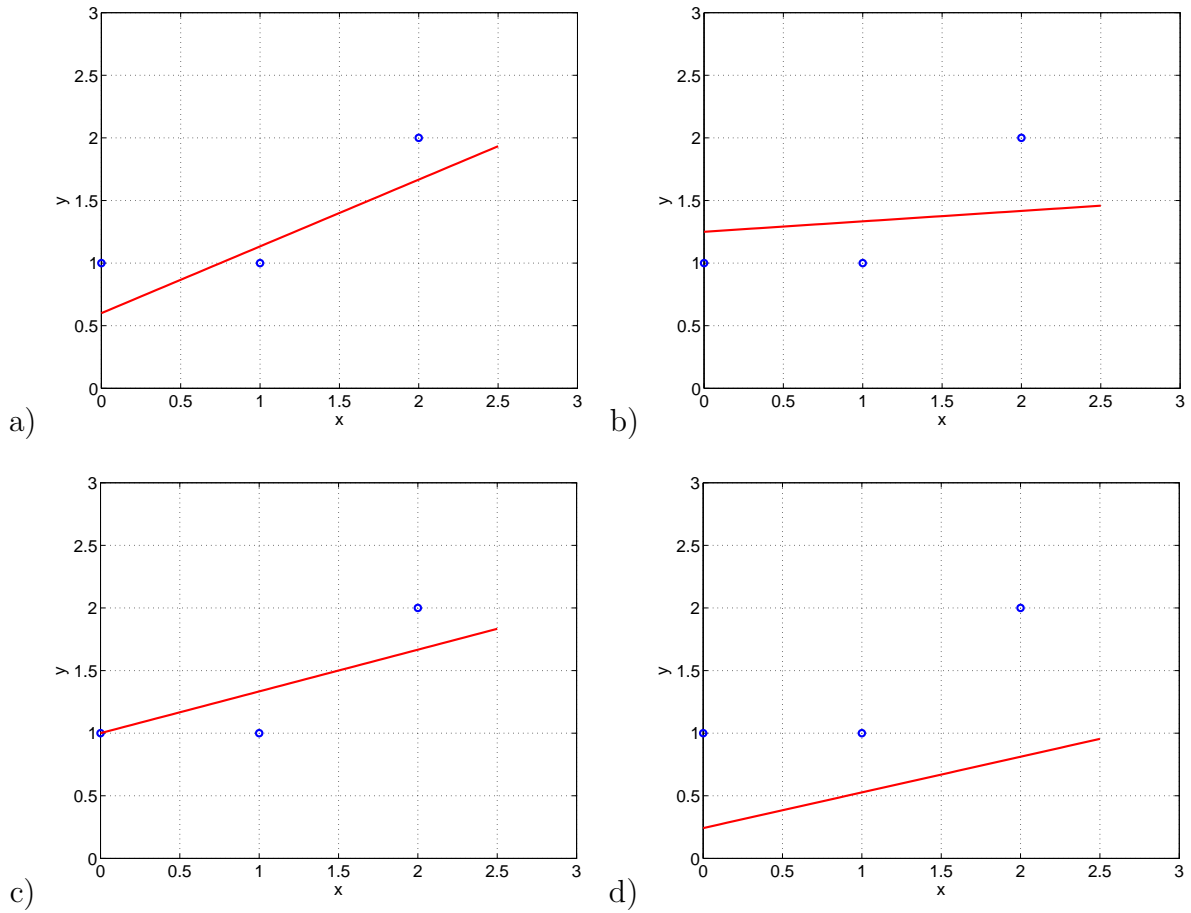


Figure 1: Plots of linear regression results with different types of regularization

## Problem 1

Figure 1 plots linear regression results on the basis of only three data points. We used various types of regularization to obtain the plots (see below) but got confused about which plot corresponds to which regularization method. Please assign each plot to one (and only one) of the following regularization method.

1.1 (2 points)  $\sum_{t=1}^3 (y_t - \theta x_t - \theta_0)^2 + \lambda \theta^2$  where  $\lambda = 1$

1.2 (4 points)  $\sum_{t=1}^3 (y_t - \theta x_t - \theta_0)^2 + \lambda \theta^2$  where  $\lambda = 10$

Briefly explain why

1.3 (2 points)  $\sum_{t=1}^3 (y_t - \theta x_t - \theta_0)^2 + \lambda(\theta^2 + \theta_0^2)$  where  $\lambda = 1$

1.4 (2 points)  $\sum_{t=1}^3 (y_t - \theta x_t - \theta_0)^2 + \lambda(\theta^2 + \theta_0^2)$  where  $\lambda = 10$

## Problem 2

We are trying to solve a regression problem with kernel linear regression models using different degree polynomial kernels. Our regression problem is a little unusual in the sense that the training input points are 1-dimensional and fixed,  $x_1, \dots, x_n$  (all distinct). Our task is to find the underlying function values at the same points and specifically at  $x_1$ . The underlying function is  $f^*(x) = E\{y|x\}$ , where the expectation is over the underlying distribution (pdf)  $p(y|x)$  governing how  $y$  depends probabilistically on  $x$ . We have no knowledge of  $f^*(x)$  or  $p(y|x)$  beyond real valued training responses  $y_1, \dots, y_n$ , sampled from  $p(y|x)$  at the training inputs.

Let's assume that our linear regression model (not in the kernel form) is given by

$$f(x; \theta, \theta_0) = \theta^T \phi(x) + \theta_0$$

where  $\phi(x)$  is the feature vector corresponding to our choice of the kernel function. We will estimate the parameter  $\theta$  and  $\theta_0$  (or  $\alpha$  and  $\theta_0$  in a kernel form) by minimizing the mean squared prediction error without regularization:

$$\frac{1}{n} \sum_{i=1}^n \left( y_i - f(x_i; \theta, \theta_0) \right)^2$$

We will then use  $f(x_1; \hat{\theta}, \hat{\theta}_0)$  as an estimator of  $f^*(x_1)$ . In other words, all we care about is the prediction at  $x_1$ . **Assume that  $n = 3$ .**

2.1 (**3 points**) Write down an expression for the bias of this estimator. Your expression should involve just  $f(x_1; \hat{\theta}, \hat{\theta}_0)$ ,  $E\{\cdot\}$ , and  $f^*(x_1)$ , as well as an explanation for what the expectation is over.

2.2 (**2 points**) Which degree polynomial kernel would we need to get zero training error, i.e., fit the three training responses perfectly?

2.3 (**2 points**) Would we get an unbiased estimator at  $x_1$  if we achieve zero training error (**Y/N**)?

2.4 (**3 points**) Suppose the noise variance at  $x_1$  is  $E\{(y_1 - f^*(x_1))^2\} = \sigma^2$ . What is the variance of our “zero training error estimator”, again at  $x_1$ ?

### Problem 3

We are trying to solve a classification problem with support vector machines. In our problem there are only a few positive training examples and we are certain that they are classified correctly. We also have a large number of negative training examples, some of which may be misclassified. We’d like to modify the basic dual form of the SVM optimization problem,

$$\begin{aligned}
 (1) \quad & \text{minimize} \quad \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j K(\mathbf{x}_i, \mathbf{x}_j) \\
 (2) \quad & \text{subject to} \quad \alpha_i \geq 0, \quad \sum_{i=1}^n \alpha_i y_i = 0
 \end{aligned}$$

to better solve this type of problem. We would like to ensure that we *won't misclassify any of the positive examples* but *could misclassify some of the negative examples*. We believe you have to introduce additional parameter(s) (or constants for the purpose of solving the quadratic programming problem) in order to achieve this.

In your solution, please use  $I^+$  to index positively labeled examples ( $y_i = +1$ ) and  $I^-$  for negative examples ( $y_i = -1$ ). In other words,  $i \in I^+$  means that  $y_i = +1$ , and  $|I^+|$  is the number of positive examples.

3.1 (6 points) Your solution must be in the dual form. You can refer to (1) and (2) above.

Minimize

subject to

3.2 (6 points) Check (Y/N) which of the following alternative criteria would work for optimizing your new parameters. We have underlined any differences between the alternatives.

- ( ) We train your SVM algorithm  $|I^+|$  times, each time leaving out one of the positive examples, and testing the classifier on the left out example. The parameter(s) are set to minimize the resulting number of misclassified examples.
- ( ) We train your SVM algorithm  $|I^-|$  times, each time leaving out one of the negative examples, and testing the classifier on the left out example. The parameter(s) are set to minimize the resulting number of misclassified examples.

Briefly explain why this would or would not work:

- ( ) We train your SVM algorithm  $n$  times, each time leaving out one of the examples, positive or negative, and testing the classifier on the left out example. The constant is set to minimize the resulting number of misclassified examples.

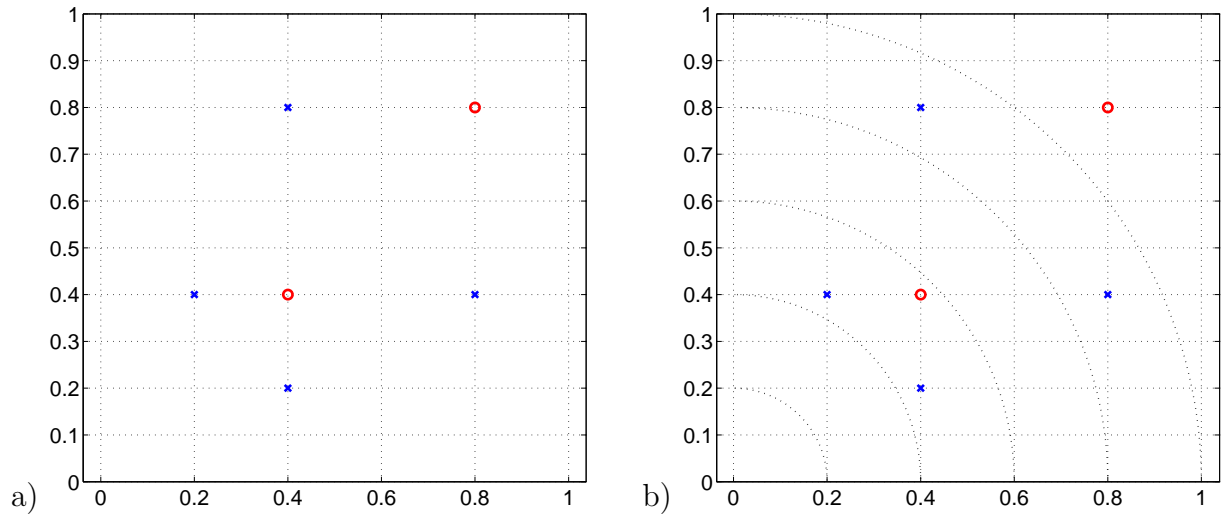


Figure 2: a) Points that should be separable with a normalized linear kernel. b) feature space with the original points overlaid with their original coordinate values.

## Problem 4

A student in a machine learning course claimed that the points in Figure 2a can be separated with “almost a linear kernel”. Hard to believe, we responded, since the points are clearly not linearly separable. But the student insisted. The “almost a linear kernel” they had in mind was the following normalized kernel:

$$K_{norm}(\mathbf{x}, \mathbf{x}') = \frac{\mathbf{x}^T \mathbf{x}'}{\|\mathbf{x}\| \|\mathbf{x}'\|}$$

4.1 (2 points) What are the feature vectors corresponding to this kernel?

4.2 (4 points) Using Figure 2b (right), graphically map the points to their new feature representation using the figure as the feature space.

4.3 (4 points) Draw the resulting maximum margin decision boundary in the feature space. Use the same Figure 2b (right). The student was right, the points are separable!

4.4 (2 points) Does the value of the discriminant function corresponding to your solution change if we scale any point, i.e., evaluate it at  $s\mathbf{x}$  instead of  $\mathbf{x}$  for some  $s > 0$ ? (Y/N)

4.5 (4 points) Draw the decision boundary in the original input space resulting from the normalized linear kernel. Use Figure 2a (left).

## Problem 5

There are many criteria for active learning. In particular, in the context of linear regression, we derived such criteria by assuming that the underlying model was also linear (in the feature space). One of the resulting criteria was based on finding points where our predictions varied the most (relative to resampled training sets from an assumed underlying model).

We will focus here on simple active learning methods for classification tasks with the perceptron algorithm. We assume that you can only ask labels for the training examples  $\mathbf{x}_1, \dots, \mathbf{x}_n$  (those we don't already have labels for). The labels are fixed once revealed so there's no reason to query the same point multiple times. The perceptron algorithm, in response to mistakes, updates its parameters according to

$$\theta \leftarrow \theta + y_t \mathbf{x}_t \quad \text{iff } y_t \theta^T \mathbf{x}_t \leq 0$$

5.1 (2 points) In our setting, would it be useful to get a label for a point that we can classify correctly? (Y/N)

5.2 (6 points) Given the current  $\theta$  we have to select which example  $\mathbf{x}_1, \dots, \mathbf{x}_n$  would be the most useful to label. Check *all* of the following criteria you believe would work as a selection criterion. We could select the point  $\mathbf{x}_t$  with

- ( ) the largest norm  $\|\mathbf{x}_t\|$
- ( ) the largest value of  $|\theta^T \mathbf{x}_t|$
- ( ) the smallest value of  $|\theta^T \mathbf{x}_t|$

Briefly explain why your chosen criterion (criteria) would work in our active learning setting:

# Additional set of figures

