Massachusetts Institute of Technology

Department of Computer Science and Electrical Engineering

6.801/6.866 Machine Vision Quiz I

Handed out: 2009 Oct. 22nd **Due on: 2009 Oct. 29th**

Problem 1a: Contour maps provide a kind of shading effect, that is, a variation of average local brightness with orientation of the surface depicted, since the spacing between adjacent contour lines depends on the slope of that surface. This may be visually helpful in interpretation of a contour map. In the following, for example, steeper areas appear darker:

Suppose that contour lines of thickness w are drawn on white paper. Let the reflectance of the ink be 0 and the reflectance of the paper 1. The map is drawn at a scale s (that is, distances in the real world are s times as large as they are in the map). Finally, h is the vertical interval between neigboring contours.

- How does the average reflectance of a region of the map depend on the slope m of the corresponding portion of the surface?
- What is the reflectance map $R(p,q)$ in this case? (Make sure that $R(p,q)$ is not negative for some gradient (p,q)).
- Is there a limit on the slope m above which there is no further variation of average reflectance with slope?

Problem 1b: Consider the surface of a rocky planet, the reflecting properties of which are modelled by Hapke's formula

$$\sqrt{\frac{\cos\theta_i}{\cos\theta_e}}$$

where θ_i is the angle between the surface normal $\hat{\mathbf{n}}$ and the direction $\hat{\mathbf{s}}$ towards the distant light source, while θ_e is the angle between the surface normal $\hat{\mathbf{n}}$ and the direction to the distant viewer — taken to be $\hat{\mathbf{z}}$ in our case.

- What do you expect a sphere of this kind of material to look like when illuminated from the same direction as it is being viewed from?
- Show that the slope m in a specific direction can be determined, if the brightness E is known, using the formula

  $$m = (E^2 - \cos\theta_s)/\sin\theta_s$$

  were θ_s is the angle between the direction to the light source ($\hat{\mathbf{s}}$) and the direction to the viewer ($\hat{\mathbf{z}}$). What happens when $\theta_s = 0$?
- Why are there spatial variations in brightness in the image below of the (almost) full moon)?

Problem 2a: It is often useful to perform "sanity checks" after obtaining the solution of a problem. For example, in the case of the shape from shading problem, we might want to check that the image irradiance equation is satisfied by the solution, that is, that $E(x, y) = R(p, q)$ along all characeristic strips.

The characteristic strip equations for solving the shape from shading problem are

$$\frac{dx}{ds} = R_p, \quad \frac{dy}{ds} = R_q, \quad \frac{dz}{ds} = pR_p + qR_q,$$
$$\frac{dp}{ds} = E_x, \quad \frac{dq}{ds} = E_y.$$

Verify that along the characteristic strip, $E(x, y) - R(p, q)$ is constant, that is

$$\frac{d}{ds}\big(E(x, y) - R(p, q)\big) = 0$$

(Hint: Use the chain rule). Conclude that, if $E(x, y) = R(p, q)$ at the beginning of the strip, then $E(x, y) = R(p, q)$ all along the strip.

Problem 2b: We have spent some time considering the possible ambiguity of shaded images. That is, we have studied the question of uniqueness of the solution of the shape from shading problem. Here we address the complementary question of the existence of a solution.

We want to determine whether certain images are "impossible" in the sense that they could not have arisen from shading on any surface. That is, no object with specified uniform surface reflectance properties could yield such an image.

(a) Show that the reflectance map of a Lambertian surface illuminated from the same direction as the viewing direction is

$$R(p, q) = \frac{1}{\sqrt{1 + p^2 + q^2}}$$

(b) Now consider an image that is uniformly bright, with brightness equal to one, outside a compact simply connected region S say. What is the shape of the surface outside the region S?

(c) Now consider the case when the image brightness is less than one *everywhere* inside the region S. What does that imply about the surface orientation in the region S?

(d) Show that there is no surface with continuous first derivatives that can give rise to such an image. (Hint: Consider extrema of the surface within the region S).

Problem 3: Here we explore the convenience of vector notation for reducing the size of expressions encountered when dealing with motion vision problems.

Let us revisit the recovery of camera motion and/or the orientation of a planar surface from an image sequence. The equation $aX + bY + cZ = d$ applies to a planar surface.

(a) Show that the equation of the plane can be written in the form

$$\mathbf{R} \cdot \mathbf{n} = 1,$$

for some vector $\mathbf{n}$, where $\mathbf{R} = (X, Y, Z)^T$. Give an expression for the unit surface normal (in terms of a, b, c, and d). What is the perpendicular distance of the plane from the origin?

(b) Suppose the plane is in front of an imaging system. Show that under perspective projection

$$\frac{1}{f}\mathbf{r} \cdot \mathbf{n} = \frac{1}{\mathbf{R} \cdot \hat{\mathbf{z}}}$$

where f is the principal distance of the imaging system, and $\mathbf{r} = (x, y, f)^T$ is the image of the point $\mathbf{R} = (X, Y, Z)^T$.

(c) Suppose now that the plane is moving with velocity $\mathbf{t} = (U, V, W)^T$ with respect to the camera (or equivalently that the camera is moving with velocity $-\mathbf{t}$ w.r.t to plane). Differentiate the perspective projection equations

$$x/f = X/Z \quad \text{and} \quad y/f = Y/Z$$

to obtain expressions for the motion field components $u = dx/dt$ and $v = dy/dt$ in terms of the components of the translational motion vector $U = dX/dt$, $V = dY/dt$ and $W = dZ/dt$.

(d) Show that the brightness change constraint equation

$$uE_x + vE_y + E_t = 0$$

leads to an equation of the form

$$E_t + (\mathbf{s} \cdot \mathbf{t})(\mathbf{r} \cdot \mathbf{n}) = 0,$$

in the case of the planar surface. Write the vector $\mathbf{s}$ in terms of components of the image brighness gradient (E_x, E_y).

(e) If $\mathbf{t}$ were known, how would you recover $\mathbf{n}$? (Hint: Consider a least squares approach using all of the image data).

(f) If $\mathbf{n}$ were known, how would you recover $\mathbf{t}$? (Hint: Consider a least squares approach using all of the image data).

Problem 4: Scanning electron microscopes produce images that are visually easy to interpret because they exhibit shading:

In a simple model of brightness in a Scanning Electron Microscope we have a reflectance map

$$R(p,q) = (1 + p^2 + q^2)/4.$$

At a singular point, $E_x = 0$ and $E_y = 0$. Assume the singular point is at $(0,0)$. Suppose that the surface near the singular point can be approximated by:

$$z = z_0 + ex + fy + ax^2 + 2bxy + cy^2$$

(a) First, show that $e = 0$ and $f = 0$ at the singular point.

(b) Using the image irradiance equation $E(x,y) = R(p,q)$ and differentiating twice with respect to x and y, derive three relationshipes between the unknown parameters a, b, and c to brightness derivatives that can be estimated from the image.

(c) By combining these relations, or suitable multiples thereof, derive the following equation, which is linear in the unknowns a, b and c:

$$b(E_{xx} - E_{yy}) = (a - c)E_{xy}$$

(d) Show that

$$E_{xx} - 2E_{xy} + E_{yy} = k\big((a-b)^2 + (b-c)^2\big)$$

for some k. What is the value of k?

(e) If $E_{xx} = E_{xy} = E_{yy} = 2$, what are a, b, and c in the series expansion given above for the surface height near the singular point? Is the answer unique?

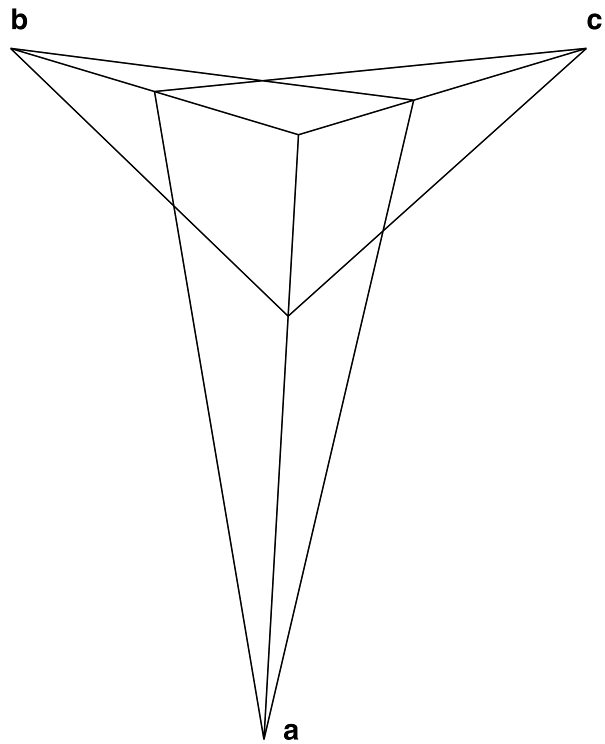
Problem 5: In this problem we study the projections of lines in the 3-D world into lines in the 2-D image. In the case of a set of parallel lines in 3-D, we find that in general the 2-D projections intersect in a point called a vanishing point.

Vanishing points are important in photogrammetry for recovering parameters of the camera and the objects being viewed. This is particularly useful for man-made objects which often have surfaces that intersect in sets of parallel edges (if we ignore the Stata Center for the moment).

We can parameterize a ray in space as

$$X = X_0 + as, \quad Y = Y_0 + bs, \quad \text{and} \quad Z = Z_0 + cs$$

where $(X_0, Y_0, Z_0)^T$ is some point on the line, $(a, b, c)^T$ the direction of the line and s a parameter that varies along the line.



(a) Where in the image is the vanishing point of the line?

(b) Show that if we pick the point on the line such that $Z_0 = 0$, then

$$\frac{x}{f} = \frac{a}{c} + \frac{X_0}{cs} \quad \text{and} \quad \frac{y}{f} = \frac{b}{c} + \frac{Y_0}{cs}$$

(c) As $s \to 0$, the image point moves to infinity. What direction in the image does it depart in?

(d) If a point moves at constant speed along the line in 3D, how does the speed of the corresponding image point vary?