Homework 8 Solutions

Problem 1.

For any $n \times n$ matrix A, and any n-element vector b, let concept $c_{A,b}$ denote $\{x \in R^n : x^T Ax + b^T x \geq 0\}$. Then P^2 is the concept class defined by these concepts $c_{A,b}$. We show that the VC-dimension of P^2 is $\binom{n}{2} + 2n$.

We first show that the VC-dimension of P^2 is at most $\binom{n}{2} + 2n$. In Homework 4, we showed that if G is a family of real-valued functions defined on a set X, and G forms a vector space of dimension d, then the VC-dimension of the class of concepts associated with G having the form $c_g = \{x \in X : g(x) \geq 0\}$ is at most d. Thus, if we show that the family of functions $x^T Ax + b^T x$ forms a vector space of dimension $\binom{n}{2} + 2n$, then we have shown the desired upper bound. If $A = (a_{ij})$, $b = (b_1, b_2, \ldots, b_n)^T$, and $x = (x_1, x_2, \ldots, x_n)^T$, then $x^T Ax + b^T x = \sum_i a_{ii} x_i^2 + \sum_{i<j} (a_{ij} + a_{ji}) x_i x_j + \sum_i b_i x_i$. It is easy to show that this family of functions forms a vector space of dimension $\binom{n}{2} + 2n$, with basis $B = \{x_i^2 : 1 \leq i \leq n\} \cup \{x_i x_j : 1 \leq i < j \leq n\} \cup \{x_i : 1 \leq i \leq n\}$.

We now show that the VC-dimension of P^2 is at least $\binom{n}{2} + 2n$. We give a set of $\binom{n}{2} + 2n$ points in R^n which can be shattered by P^2. We consider points $u^{(i)}$ and $v^{(i)}$, (for $1 \leq i \leq n$), and points $z^{(i,j)}$, (for $1 \leq i < j \leq n$) which are defined as follows. Each $u^{(i)}$ is 0 in all dimensions except the i-th, where it is 1, and each $v^{(i)}$ is 0 in all dimensions except the i-th, where it is -1. Each $z^{(i,j)}$ is 0 in all dimensions except the i-th and j-th where it is 1. For example, $u^{(1)} = (1, 0, 0, \ldots, 0)^T$, $v^{(1)} = (-1, 0, 0, \ldots, 0)^T$, and $z^{(1,2)} = (1, 1, 0, 0, \ldots, 0)^T$. Now, the value of $x^T Ax + b^T x$ for each of these points is:

$$\begin{aligned} a_{ii} + b_i, &\text{ for } u^{(i)} \\ a_{ii} - b_i, &\text{ for } v^{(i)} \\ a_{ii} + a_{ij} + a_{ji} + a_{jj} + b_i + b_j, &\text{ for } z^{(i,j)} \end{aligned}$$

Notice that any weight a_{ii} or b_i affects the label of only examples $u^{(i)}$, $v^{(i)}$, and the $z^{(i,j)}$ $(i < j \leq n)$. Thus, we can achieve any labelling of the $u^{(i)}$ and $v^{(i)}$ as follows, without affecting the labels of points other than the $z^{(i,j)}$:

$$\begin{aligned} a_{ii} = 1, b_i = 0, &\quad \Rightarrow \quad u^{(i)}, v^{(i)} \text{ are positive examples} \\ a_{ii} = 0, b_i = 1, &\quad \Rightarrow \quad u^{(i)} \text{ is positive, } v^{(i)} \text{ is negative} \\ a_{ii} = -1, b_i = 0, &\quad \Rightarrow \quad u^{(i)}, v^{(i)} \text{ are negative examples} \\ a_{ii} = 0, b_i = -1, &\quad \Rightarrow \quad u^{(i)} \text{ is negative, } v^{(i)} \text{ is positive} \end{aligned}$$

Once we have set the weights of a_{ii}, a_{jj}, b_i and b_j, then the label of $z^{(i,j)}$ depends on the weights a_{ij} and a_{ji}. Moreover, the values of weights a_{ij} and a_{ji} affect the label of only one example $z^{(i,j)}$. Since the weights of a_{ii}, a_{jj}, b_i and b_j were chosen to have absolute value at most 1, choosing $a_{ij} = -5, a_{ji} = 0$ makes $z^{(i,j)}$ negative, and choosing $a_{ij} = 5, a_{ji} = 0$ makes $z^{(i,j)}$ positive. Thus, we have shown a set of $\binom{n}{2} + 2n$ points such that for any labelling of these points there is a concept in P^2 consistent with it.

Thus, the VC-dimension of the concept class P^2 is $\binom{n}{2} + 2n$.