Homework 4

Homework rules:

- Homeworks are due one week after they are handed out, unless otherwise specified. *No late homeworks accepted!*
- Feel free to talk to other members of the class in doing your homework. I'm more concerned that you learn how to solve the problem, than that you demonstrate that you solved it entirely on your own. You should, however, write up your solution yourself. Please try to keep solutions brief and clear. You may refer to the literature, but do not refer to handouts from previous years' classes (in case I repeat some problems from previous years).

Due Wednesday, October 12

Problem 1.

Let G denote a family of real-valued functions defined on a set X. Suppose that G forms a vector space of dimension d. (That is, there is a set of d "basis functions" such that any element of G can be expressed as a linear combination of the basis functions using real coefficients.) For example, X may be the reals and G the set of polynomials of degree at most $d-1$. Or, X may be a real d-dimensional space and G the set of linear combinations of the d coordinates (thus defining hyperplanes). We want to show that the VC-dimension of the class of concepts associated with G having the form $c_g = \{x \in X : g(x) \geq 0\}$ is at most d.

Pick any set $S = \{x_1, x_2, \ldots, x_{d+1}\}$ of $d+1$ distinct points in X. We want to show that S is not shattered by the concept class associated with G.

(a) For each $g \in G$, define the vector $v_g = (g(x_1), g(x_2), \ldots, g(x_{d+1}))$. Here each v_g has length $d+1$. Argue that the set $V = \{v_g : g \in G\}$ is a linear subspace of R^{d+1} of dimension at most d.

(b) Argue that there must exist a nonzero vector $w \in R^{d+1}$ that is orthogonal to V. Argue that without loss of generality w contains at least one negative component.

(c) Let P denote the set $\{i : w_i \geq 0\}$, and let N denote the set $\{i : w_i < 0\}$. (N is nonempty by (b).) Argue that for any $g \in G$ we have

$$\sum_{i \in P} w_i g(x_i) = \sum_{i \in N} (-w_i) g(x_i) \ .$$

(d) Argue that no g in G can be nonnegative on $\{x_i : i \in P\}$ and negative on $\{x_i : i \in N\}$. Therefore S is not shattered and the concept class associated with G has VC-dimension at most d.