1 Approximation Algorithms

NP-completeness.
Recall definition of approximation algorithms.

The Probabilistic Method—Value of Random Answers

Idea: to show an object with certain properties exists

- generate a random object
- prove it has properties with nonzero probability
- often, "certain properties" means "good solution to our problem"
- random routing as example

Max-Cut:

- Define
- NP-complete
- Approximation algorithms
- factor 2
- "expected performance," so doesn't really fit our RP/ZPP framework
- but does show such a cut **exists**

Set balancing.

- minimize max bias.
- $4\sqrt{n \ln n}$.
- Spencer—10 lectures on the probabilistic method

The Probabilistic Method for Expectations

Outline

- goal to show exists object of given "value"
- give distribution with greater "expected value"
- deduce goal

MAX SAT

Sometimes, it's hard to get hands on a good probability distribution of random answers. Define.

- literals
- clauses
- NP-complete

random set

- achieve $1 - 2^{-k}$
- very nice for large k, but only 1/2 for $k = 1$

LP

$$\begin{aligned} \max & \quad \sum z_j \\ \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_1) &\geq z_j \end{aligned}$$

Analysis

- $\beta_k = 1 - (1 - 1/k)^k$. values $1, 3/4, .704, \ldots$
- Random round y_i
- Lemma: k-literal clause sat w/pr at least $\beta_k \hat{z}_j$.
- proof:
  - assume all positive literals.
  - prob $1 - \prod(1 - y_i)$
  - maximize when all $y_i = \hat{z}_j/k$.
  - Show $1 - (1 - \hat{z}/k)^k \geq \beta_k \hat{z}_k$.
  - at $z = 0, 1$ these two sides are equal
  - in between, right hand side is linear
  - first deriv of LHS is $(1 - z/k)^k$, second deriv is $-(1 - 1/k)(1 - z/k)^{k-2} < 0$,
  - so LHS cannot cross below and then return, must always be above RHS
- Result: $(1 - 1/e)$ approximation (convergence of $(1 - 1/k)^k$)
- much better for small k: i.e. 1-approx for $k = 1$

LP good for small clauses, random for large.

- Better: try both methods.

- n_1, n_2 number in both methods
- Show $(n_1 + n_2)/2 \geq (3/4) \sum \hat{z}_j$
- $n_1 \geq \sum_{C_j \in S^k} (1 - 2^{-k}) \hat{z}_j$
- $n_2 \geq \sum \beta_k \hat{z}_j$
- $n_1 + n_2 \geq \sum (1 - 2^{-k} + \beta_k) \hat{z}_j \geq \sum \frac{3}{2} \hat{z}_j$

Wiring

Problem formulation

- $\sqrt{n} \times \sqrt{n}$ *gate array*
- Manhattan wiring
- boundaries between gates
- fixed width boundary means limit on number of crossing wires
- optimization vs. feasibility: minimize max crossing number
- focus on single-bend wiring. two choices for route.
- Generalizes if you know about multicommodity max-flow

Linear Programs, integer linear programs

- Black box
- Good to know, since great solvers exist in practice
- Solution techniques in other courses
- LP is polytime, ILP is NP-hard
- LP gives hints—rounding.

IP formulation

- x_{i0} means x_i starts horizontal, x_{i1} vertical
- $T_{b0} = \{i \mid \text{net } i \text{ through } b \text{ if } x_{i0}\}$
- T_{b1}
- IP

$$\begin{array}{rcl} \min & & w \\ x_{i0} + x_{i1} & = & 1 \\ \sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} & \leq & w \end{array}$$

Rounding

- Solution $\hat{x}_{i0}$, $\hat{x}_{i1}$, value $\hat{w}$.
- rounding is Poisson vars, mean $\hat{w}$.
- For $\delta < 1$ (good approx) $\Pr[\geq (1+\delta)\hat{w}] \leq e^{-\delta^2 \hat{w}/4}$
- need $2n$ boundaries, so aim for prob. bound $1/2n^2$.
- solve, $\delta = \sqrt{(4 \ln 2n^2)/\hat{w}}$.
- So absolute error $\sqrt{8\hat{w} \ln n}$
  - Good ($o(1)$-error) if $\hat{w} \gg 8 \ln n$
  - Bad ($O(\ln n)$ error) if $\hat{w} = 2$ (invoke other chernoff bound)
  - General rule: randomized rounding good if target logarithmic, not if constant

Generalize

- Multicommodty flow generalization
- Same rounding works