Problem Set 3

Due: Wednesday, September 28, 2011.

Collaboration policy: collaboration is strongly encouraged. However, remember that

1. You must write up your own solutions, independently.

2. You must record the name of every collaborator.

3. You must actually participate in solving all the problems. This is difficult in very large groups, so you should keep your collaboration groups limited to 3 people in a given week.

4. No bibles. This includes solutions posted to problems in previous years.

Problem 1. Which of the following statements about flows are true and which are false? Justify your answer with a (short) proof or counterexample. (Here “raw flow” refers to the “gross flow” description Prof Karger used in class. Raw flows on \((x,y)\) and \((y,x)\) are both nonnegative, less than the corresponding capacities, as opposed to net flows where \(f(x,y) = -f(y,x)\).)

(a) In any maximum flow, and for all vertices \(v\) and \(w\), either the raw flow from \(v\) to \(w\) or the raw flow from \(w\) to \(v\) is 0.

(b) There always exists a maximum flow such that, for all vertices \(v\) and \(w\), either the raw flow from \(v\) to \(w\) or the raw flow from \(w\) to \(v\) is 0.

(c) If all directed edges in a network have distinct capacities, then there is a unique maximum flow.

(d) If we replace each directed edge in a network with two directed edges in opposite directions with the same capacity and connecting the same vertices, then the value of the maximum flow remains unchanged.

(e) If we add the same positive number \(\lambda\) to the capacity of every directed edge, then the minimum cut (but not its value) remains unchanged.

(f) Flow is transitive: if in a graph there is a flow of value \(v\) from \(s\) to \(t\), and there is a flow of value \(v\) from \(t\) to \(u\), then there is a flow of value \(v\) from \(s\) to \(u\).

Problem 2. Critical edges for maximum flow.
(a) An edge is *upward critical* if increasing its capacity increases the maximum flow value. Does every network have an upward-critical edge? Give an algorithm to identify all upward-critical edges in a network. Its running time should be substantially better than that of solving $m$ maximum-flow problems.

(b) An edge is *downward critical* if decreasing its capacity (by any amount) decreases the maximum flow value. Is the set of upward-critical edges the same as the set of downward-critical edges? Describe an algorithm for identifying all downward-critical edges, and analyze your algorithm’s worst-case complexity.

**Problem 3.** The minimum flow problem is a close relative of the max flow problem with nonnegative lower bounds on edge flows (if an edge $(i,j)$ has a lower bound $l_{ij} \geq 0$, then $f_{ij}$ must satisfy $l_{ij} \leq f_{ij} \leq u_{ij}$). In the minimum flow problem we wish to send a minimum amount of flow from the source $s$ to the destination $t$ while satisfying given lower and upper bounds on edge flows $f_{ij}$.

(a) Show how to solve the minimum flow problem by using two applications of any maximum flow algorithm that applies to problems with zero lower bounds on edge flows (e.g., the algorithms described in the lecture). **Hint:** first construct any feasible flow and then convert it into a minimum flow.

(b) Prove the following min-flow-max-cut theorem. Let the lower bound on the cut capacity of an $s$-$t$ cut $(S,T = V \setminus S)$ be defined as $\sum_{(i,j) \in S \times T} l_{ij} - \sum_{(i,j) \in T \times S} u_{ij}$. Show that the minimum value of all feasible flows from node $s$ to node $t$ equals to the maximum lower bound on cut capacity of all $s$-$t$ cuts.

(c) A group of students wants to minimize their lecture attendance by sending only one of the group to each of $n$ lectures. Lecture $i$ begins at time $a_i$ and ends at time $b_i$. It requires $r_{ij}$ time to commute from lecture $i$ to lecture $j$. Do not assume these times are integers. Develop a flow-based algorithm for identifying the minimum number of students needed to cover all the lectures. **Hint:** reduce to minimum flow.

**Problem 4.** At a certain point in the season, each team $i$ in the American League has won a certain number $w_i$ of games, and there remain $q_{ij}$ games to be played between teams $i$ and $j$. Assuming no ties or cancelled games, develop an efficient algorithm for deciding if the Red Sox can still win the league pennant, i.e. whether a particular team can still win the most games after all games have been played.

**Problem 5.** It is visit day for the new graduate admits, and each wants a one-on-one visit with a certain set of faculty members. The day will be divided into time slots during which each student can meet at most one faculty member, and vice versa.
(a) Suppose that the numbers of faculty and students are equal, each student wants to meet exactly $d$ faculty, and each faculty member is on the request list of $d$ students. Conclude that one can schedule a single slot in which every student is meeting someone. **Hint:** show the minimum cut of the desired-meetings graph is large.

(b) Conclude that it is possible to schedule all the meetings to take place in $d$ time slots.

(c) Consider an arbitrary set of desired meetings. Obviously one needs at least as many slots as there are faculty to meet a given student, and students to meet a given faculty. Prove that one can arrange all meetings with no more slots than this number $s$.

(d) Show that the schedule can be computed in $O(s m \sqrt{n})$ time.

Note: this algorithm is actually used to schedule grad visit day, but some extra complexities need to be factored in (specifically, the fact that some faculty are unavailable at certain times), making the problem NP-hard.

**OPTIONAL Problem 6.** Arbitrary augmenting paths are guaranteed to terminate in finite time with a maximum flow only if the edge weights are rational.

(a) Give a graph with non-rational capacities on which some augmenting path algorithm fails to terminate in finite time (even if each augmenting path is fully augmented).

(b) (Harder) Give a graph on which some augmenting path algorithm fails to terminate it finite time, and whose limiting flow is not maximum.