6.854 Guest	Lecture 5	Sept. 17, 2007

Universal hashing: (recall)

polynomial-size 2-universal hash family:

- $\Pr\{\text{two elements collide}\} = \frac{1}{n}$

$\Rightarrow E[\#\text{collisions with an element}] = O(1)$

- e.g. $x \mapsto (ax+b \bmod p) \bmod n$

$x \in [1,m]$, a, b random, p prime in $[m, 2m]$

- but max load can be $\Theta(\sqrt{n})$

Perfect hashing: hashing with no collisions

- perfect hash function has no collisions on key set
- random hash function on n keys likely collides

(n! perfect permutations, n^n functions, ratio $= \Theta\left(\frac{\sqrt{n}}{e^n}\right)$)

- what about random hash functions on $s < n$ keys?
  - $E[\#\text{collisions}] = \binom{s}{2} \cdot \frac{1}{n} < \frac{1}{2} s^2 n$
  - Markov's Inequality: $\Pr\{X \geq x\} \leq E[X]/x$

  $\Rightarrow \Pr\{\#\text{ collisions} \geq 1\} < \frac{1}{2} s^2 n$

  $\leq \frac{1}{2}$ if $s = \sqrt{n}$.

  $\Rightarrow O(1)$ expected trials before success if $s = \sqrt{n}$

- Birthday Paradox shows optimality:
  - s samples of $\{1, 2, \ldots, n\}$

  $\Rightarrow \Pr\{\text{collision}\} = \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \cdots \cdot \left(1 - \frac{s}{n}\right)$

  $\approx e^{-1/n - 2/n - \cdots - s/n} \approx e^{-\frac{1}{2} s^2/n}$

  $\Rightarrow$ when $s = \sqrt{n}$, $\Theta(1)$ chance of collision

  - 23 for birthdays (even if independent)

2-level perfect hashing: reducing to linear space

- so far have quadratic-space perfect hashing: $s = \sqrt{n}$ keys in size-n table
- hash n keys into table of size n using 2-universal hashing ($\Rightarrow$ collisions)
- build quadratic-size perfect hash table on keys in each bucket b_i (randomly + repeat)

$\Rightarrow$ total space $= \sum_{k=1}^{n} b_k^2 = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} [i \in b_k] \right)^2 = \sum_{i=1}^{n} \left(b_i + 2 \sum_{j=1}^{s} \underbrace{C_{ij}}_{\text{collision}} \right)$

$\Rightarrow E[\text{total space}] = n + 2 \cdot E[\text{total \# collisions}] = O(s)$

- Markov Inequality $\Rightarrow \Pr\{\text{space} \geq c \cdot n\} \leq \frac{1}{2}$ for some c

$\Rightarrow O(1)$ expected trials until $\sum b_k^2 = O(n)$

- then build perfect hash table on each bucket using 2-universal hashing ($O(1)$ expected trials/bucket)

- expected linear time, guaranteed linear space
- <u>easy</u>: $6n$ cells of space
- <u>hard</u>: $n + o(n)$ cells of space [<u>succinct</u> data str.]

- <u>Las Vegas</u> algorithm: guaranteed correct, just time bound guarantee is probabilistic
- vs. <u>Monte Carlo</u> algorithm: answer correct with probability $\geq 2/3$, say

Derandomization:

- only m^2 top-level 2-universal hash functions
- try them all
- ditto for bottom-level hash functions

$\Rightarrow$ time polynomial in m, not n

Dynamic:

- top-level 2-universal hash is dynamic
- allow for growth in bottom-level perfect hash by factor of 2 (up to $2b_k \Rightarrow 4b_k^2$ space)
- choose perfect hash for new element randomly
- execution as if element there in first place
- if collision, rebuild entire bucket
- if bucket overflows, rebuild 2× larger
- watch if $\sum b_k^2$ grows too big

$\Rightarrow$ rebuild entire structure

Dijkstra's shortest-paths algorithm (recall)

- priority queue on vertices, keyed on distance estimate from source
- keys only decrease → m decrease-key's
- min only increases → n delete-min's

  MONOTONICITY PROPERTY
- standard run time:
  - $O(m \lg n)$ via binary heap
  - $O(m + n \lg n)$ via Fibonacci heap

Bucketing:

- often edge weights are small integers say in $\{1, 2, \ldots, C\}$

⇒ keys in $\{0, 1, \ldots, (n-1)C\}$ $(\leq nC)$

- if m is large, many equal keys (Pigeonhole)
- idea: keep equal keys together in a bucket
- store heap on nC buckets
- but only $C+1$ buckets $\{x, x+1, \ldots, x+C\}$ active at any time

⇒ $O(m \lg C)$ via binary heaps (& $O(C)$ space)

- $O(m + n \lg C)$ via Fibonacci heaps
- $O(m + nC)$ via simple array: [Dial 1969]
  - circular array of size $C+1$
  - pointer to current key x
  - advance pointer to next nonempty bucket
  - advance $\leq nC$ times by monotonicity

2-level bucketing:

- blocks of b consecutive keys
- $\Rightarrow nC/b$ blocks
- array of nC/b block summaries (item counts)
- insert(x) updates block x div b & its summary
- decrease-key(x) ditto.
- delete-min scans summaries for nonempty block, then scans block
  - monotonicity $\Rightarrow \leq nC/b$ total advancing through summaries
  - $\leq b$ to scan block (coallescing like keys)
- $\Rightarrow n$ delete-min's cost $nC/b + nb$
- $\Rightarrow$ minimized when $C/b = b$ i.e. $b = \sqrt{C}$
- $\Rightarrow O(m + n\sqrt{C})$ time for shortest paths
- as before, only $C+1$ active keys
  $\Rightarrow$ only need $O(C)$ space
- avoid cost of array initialization by re-using $C/b = \sqrt{C}$ blocks

3-level bucketing:

- blocks, superblocks, summary
- block size $C^{1/3}$
- $\Rightarrow O(m + nC^{1/3})$ time
- can we go to $O(m+n)$?
  NO: insert cost rises

k-level bucketing (tries)

- depth-k tree over key space of nC

$\Rightarrow$ branching factor of $\Delta = (C+1)^{1/k}$
  except at top where it's $n\Delta$

- insert, decrease-key: $O(k)$ time
- delete-min: $O(k\Delta)$ to navigate lower levels
  $+\ O(n\Delta)$ overall to advance top level

$\Rightarrow$ total cost: $O(mk + nk\Delta)$

$\Rightarrow$ minimized when $m = n\Delta$ i.e. $C = (m/n)^k$
  i.e. $k = \lg C / \lg \frac{m}{n}$

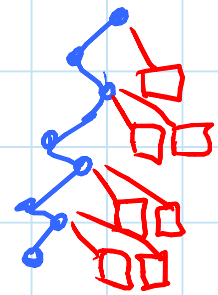
$\Rightarrow$ $O(m \log_{m/n} C)$ time

- space $= kn = O(n \log_{m/n} C)$

$\boxed{\text{Q:}}$ $nk\Delta$ vs. $n\Delta$ seems sloppy...
is a different division better?
(by say $\lg\lg$ factor)

Lazy version: [Denardo & Fox 1979]

- just one active root-to-leaf path to current minimum
- for each branch off of active path, accumulate linked list of stuff



- push stuff down "as necessary"

- insert(x):
  - walk down tree
  - if fall off active path: add x to list
  - else: add to leaf block
- decrease-key(x):
  - remove from current list
  - descend active path until fall off (can't go left of it by monotonicity)
  - add to list
- delete-min:
  - remove item from bottom of active path
  - walk up to first nonempty list hanging off right
  - expand & push elements down to form new active path (& find new min)

Analysis:

- each element descends k times
- $\Rightarrow$ total descent cost (pushing down) $\leq n \cdot k$.
- insert costs $O(k)$ (and pays for future descent)
- decrease-key pays $O(1)$ to go up 1, remaining $O(k)$ charged to descent $\Rightarrow O(1)$ am.
- delete-min: $O(k)$ to rise, descent free, $O(C^{1/k})$ to scan leaf block & up-down transition block
- $\Rightarrow O(m + n(k + C^{1/k}))$ time
- $\Rightarrow$ minimized when $k = C^{1/k}$ i.e. $C = k^k$ i.e. $k = \lg C / \lg\lg C$
- $\Rightarrow O(m + n \lg C / \lg\lg C)$ time — better than Fibonacci!
- $O(n + k\,C^{1/k})$ space ~ typically $O(n)$

Further improvements:

- Heap On Top (HOT) queues: $O(m + n \lg^{1/3} C)$ time [Cherkassy, Goldberg, Silverstein – SICOMP 1999]
- van Emde Boas [Lecture 6]: $O(m \lg\lg(nC))$ time