6.854 Advanced Algorithms

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Farkas' Lemma and Duality

10.1 Introduction

Let us recap the definitions we saw in the last lecture.

Definition 1 (Vertex of a Polyhedron) *Given a polyhedron $\mathcal{P}$, a point $p \in \mathcal{P}$ that is uniquely optimal for some objective c is called a vertex of $\mathcal{P}$.*

Definition 2 (Extreme Point of a Polyhedron) *A point $p \in \mathcal{P}$ that is not a convex combination of $y, z \in \mathcal{P}, y, z \neq p$ is called an extreme point of the polyhedron.*

Definition 3 (Basic Feasible Solution) *For an n-dimensional LP, a point $p \in \mathcal{P}$ at which n linearly independent constraints are tight is called a basic feasible solution.*

In the last lecture, we proved that p is a vertex $\Rightarrow$ p is an extreme point. In this lecture, we show that the three notions defined above are equivalent. For the proof of the following claims, we assume that the LP is given in the canonical form.

1. **Extreme Point $\Rightarrow$ Basic Feasible Solution**

   An extreme point x by definition lies in the polyhedron $\mathcal{P}$. Hence it is feasible. Now, suppose x is an extreme point but not a basic solution. Set $T = \{i \mid a_i.x = b_i\}$. For $i \notin T$, $a_i.x > b_i$. Since x is not a basic solution, $\{a_i \mid i \in T\}$ does not span $\mathcal{R}^n$. So, there is a vector $d \neq 0$ such that $d.a_i = 0\ \forall i \in T$. We obtain a contradiction if we consider the points $x \pm \epsilon d$ for small ϵ. For $i \in T$, $a_i.d = 0$ and $a_i.(x \pm \epsilon d) = a_i.x = b_i$. For $i \notin T$, $a_i.x > b_i$. We claim that for sufficiently small ϵ, $a_i.(x \pm \epsilon) \geq b_i$. Indeed, choosing $\epsilon = \frac{a_i.x - b_i}{a_i.d}$ will suffice. This, in particular implies that the points $x \pm \epsilon d$ are in $\mathcal{P}$. Since $x = \frac{(x+\epsilon d)+(x-\epsilon d)}{2}$ is a convex combination of two points in $\mathcal{P}$, it is not extreme. This is a contradiction to our hypothesis that x is an extreme point. ■

2. **Basic Feasible Solution $\Rightarrow$ Vertex**

   Suppose x is a basic feasible solution. Define $T = \{i \mid a_i.x = b_i\}$. Consider the objective as minimizing $c.y$ for $c = \sum_{i \in T} a_i$.
   $c.x = \sum_{i \in T}(a_i.x) = \sum_{i \in T} b_i$. For $x' \in \mathcal{P}$, $c.x' = \sum_{i \in T}(a_i.x') \geq \sum_{i \in T} b_i$ with equality only if $a_i.x' = b_i \; \forall i \in T$. This implies that $x' = x$ and that x uniquely minimizes the objective $c.y$. ■

We have thus shown that the notions of vertices, extreme points and basic feasible solutions are equivalent.

Now we want to prove that for any bounded LP and any objective c, there is an extreme point that gives the optimum solution. Note that this immediately gives us an algorithm to solve an LP, since there are at most $\binom{m}{n}$ vertices for the polyhedron corresponding to the LP in n variables and m constraints. To solve the LP, we just have to evaluate the objective function on each of these points and choose the optimum. [1]

Theorem 1 *Any bounded LP in standard form has an optimum at an extreme point.*

Proof: Suppose x is an optimal solution to an LP. We construct a vertex v such that v is optimal.

If x is a vertex, there is nothing to prove. Suppose not. $\exists y \neq 0$ such that $x + y, x - y \in \mathcal{P}$. Therefore, $A(x + y) = A(x - y)$ and $Ay = 0$. Similarly $c(x + y) = c(x - y)$ and $cy = 0$. If all the entries of y, $y_i \geq 0$, negate y. This makes at least one $y_j < 0$. Increase $\lambda \geq 1$ until $x + \lambda y$ has a new zero entry (as compared to x). The new x is tight against the new constraint $x_i \geq 0$. Also $x_i = 0 \Rightarrow y_i = 0 \Rightarrow x'_i = 0$. This can happen only n times. After that, we will get to the extreme point. ■

A counterexample to this theorem can be constructed if the LP were in canonical form. Consider the LP minimize x such that $0 \leq x \leq 1$. This LP has an infinity of solutions but none of them is an extreme point.

10.2 Farkas' Lemma

We saw that the brute force approach to solving an LP takes exponential time.

Consider the decision vertex of an LP that asks whether the LP has a solution with $cx < \delta$. This problem is clearly in NP since we can produce a point that satisfies the conditons and the veracity of the solution can be checked in polytime. A more interesting question

[1] This is still exponential since if $m = 2n$, $\binom{m}{n} = \Omega(\frac{2^{2n}}{\sqrt{n}})$.

is whether this problem is in $coNP$. To prove this, we have to produce a succinct easily-verifiable proof to the fact that *there is no x such that $cx < \delta$.* **Farkas' Lemma** helps us do this.

Lemma 1 (Farkas' Lemma) *Exactly one of $\exists x, Ax = b, x \geq 0$ and $\exists y, yA \geq 0, yb < 0$ is true.*

Proof: Suppose $\exists x, Ax = b, x \geq 0$. Then $yA \geq 0$ implies $yAx \geq 0$.

Suppose $\not\exists x, Ax = b, x \geq 0$. Consider $\{Ax \mid x \geq 0\}$. This is clearly a convex set (a cone) that does not contain b. From the previous lecture, we know that there is a hyperplane y that separates b and $\{Ax \mid x \geq 0\}$. This means $\forall Ax, yb < yAx$. In particular, $yb < 0$ since $0 \in \{Ax \mid x \geq 0\}$. Consider the unit vectors e_i. Since $\lambda e_i \geq 0$ for $\lambda > 0$, $A(\lambda e_i)$ is in the cone. $yb < y(A\lambda e_i) = \lambda y a_i$. This is true for any $\lambda > 0$. Therefore $\lim_{\lambda \to \infty} \frac{yb}{\lambda} \leq yA_i$ implies $yA \geq 0$. ■

There is a similar Farkas' Lemma for the canonical form too. It states that exactly one of $yA \leq c$ and $\exists x \geq 0, Ax = 0, cx < 0$ holds.

10.3 Duality

Our goal here is to obtain a lower bound on $z = min\{c.x \mid Ax = b, x \geq 0\}$. We call this LP the primal LP. We multiply $Ax = b$ by some y to get $yAx = yb$. If we require that $yA \leq c$, $yb = yAx \leq cx$ and therefore, yb is a lower bound on optimum value z. Now, consider the LP $w = max\{yb \mid yA \leq c\}$ which we call the dual LP. We have just argued that $z \geq w$.

Theorem 2 (Weak Duality) *Consider $z = min\{c.x \mid Ax = b, x \geq 0\}$ and $w = max\{yb \mid yA \leq c\}$. Then $z \geq w$.*

Note that the dual of the dual LP is the primal LP.

Corollary 1 *If the primal LP is unbounded, the dual is infeasible.*

We now show that z is indeed equal to w. This is called Strong Duality.

Theorem 3 (Strong Duality) *Consider $z = min\{c.x \mid Ax = b, x \geq 0\}$ and $w = max\{yb \mid yA \leq c\}$. Then $z = w$.*

Proof: Suppose the primal is feasible and $z > w$. Recall that $w = max\{yb \mid yA \leq c\}$ implies that there is no solution to the polytope $\{yA \leq c, yb \geq c\}$. By Farkas' Lemma,

$y(A \mid -b) \leq (c \mid -z)$ implies that there exists

$$\begin{pmatrix} x \\ q \end{pmatrix} \geq 0$$

such that

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} = 0$$

and

$$\begin{pmatrix} c & -z \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} < 0$$

$x, q \geq 0, Ax = qb, cx < qz$.
Case (1) : $q > 0$. $A\frac{x}{q} = b, \frac{x}{q} \geq 0, \frac{cx}{q} < z$. $\frac{x}{q}$ is a feasible point and $\frac{cx}{q}$ is less than the claimed optimum z. Contradiction.
Case (2) : $q = 0$. $Ax = 0, cx < 0$. Consider optimum x^* such that $Ax^* = b, x^* \geq 0, cx^* = z$. Consider $x^* + x$. $c(x^* + x) < z$. This means that z is not optimal. Contradiction again.
We conclude that the primal and dual optima are equal. ■

Is determining feasibility of an LP easier than determining optimality ? We argue that this is not the case. We construct a reduction from the problem of finding an optimal solution of LP_1 to the problem of finding a feasible solution of LP_2. LP_1 is $min\{c.x \mid Ax = b, x \geq 0\}$. Consider $LP_2 = \{Ax = b, x \geq 0, yA \leq c, cx = by\}$. Any feasible solution of LP_2 gives an optimal solution of LP_1 by Strong Duality. Finding an optimal solution is thus no harder than finding a feasible solution.