Massachusetts Institute of Technology
6.854J/18.415J: Advanced Algorithms
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Handout 11
Monday, October 6, 2003

Problem Set 6

Due: Wednesday, October 15, 2003.

NONCOLLABORATIVE Problem 1. Consider a unit-capacities network where each edge has cost 1. Does Dinitz's normal blocking-flow algorithm compute a min-cost max-flow in the graph? why or why not?

Problem 2. Suppose you are given two polyhedra $P = \{x \mid Ax \leq b\}$ and $Q = \{x \mid Dx \leq e\}$.

(a) Using our polyhedral techniques, prove that, if polyhedra P and Q have empty intersection (i.e., no point is in both), then there is a separating hyperplane for P and Q (i.e., a vector c such that $c \cdot x < c \cdot y$ for all $x \in P$ and $y \in Q$).

(b) Suppose you have an algorithm for solving linear programming. Argue that with this algorithm you can either find a point in $P \cap Q$ or find their separating hyperplane c. Hint: Use duality or Farkas's lemma on a linear program that combines P and Q.

Problem 3. Another way to formulate the maximum-flow problem as a linear program is via flow decomposition. Suppose we consider all (exponentially many) s-t paths P in the network G, and let f_P be the amount of flow on path P. Then maximum flow says to find

$$\begin{aligned} z &= \max \sum f_P, \\ \sum_{P \ni e} f_P &\leq u_e, \\ f_P &\geq 0. \end{aligned}$$

(The first constraint says that the total flow on all paths through e must be less than u_e.) Take the dual of this linear program and give an English explanation of the objective and constraints.

Problem 4. Although the dual can tell you a lot about the structure of a problem, knowing an optimal dual solution does not in general help you solve the primal problem. Suppose we had an algorithm that could optimize an LP with an $m \times n$ constraint matrix in $O((m+n)^k)$ time given an optimal solution to the dual LP.

(a) Argue that any LP optimization problem can be transformed into the following form:

$$\begin{array}{ll} \text{minimize} & 0 \cdot x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

(This LP has optimum value 0 if it is feasible, and ∞ if it is infeasible.)

(b) What is the dual of this linear program?

(c) Argue that, if the primal is feasible, then the dual has an obvious optimum solution.

(d) Deduce that, given the algorithm above, you can build an LP algorithm that will solve any LP *without* knowing a dual solution, in the same asymptotic time bounds as the algorithm above.