Problem Set 4

Due: Monday, September 29, 2003.

Problem 1. Which of the following statements about flows are true and which are false? Justify your answer with a (short) proof or counterexample.

(a) In any maximum flow, and for all vertices v and w, either the raw flow from v to w or the raw flow from w to v is 0.

(b) There always exists a maximum flow such that, for all vertices v and w, either the raw flow from v to w or the raw flow from w to v is 0.

(c) If all directed edges in a network have distinct capacities, then there is a unique maximum flow.

(d) If we replace each directed edge in a network with two directed edges in opposite directions with the same capacity and connecting the same vertices, then the value of the maximum flow remains unchanged.

(e) If we add the same positive number λ to the capacity of every directed edge, then the minimum cut remains unchanged.

Problem 2. Suppose you have already computed the maximum flow in a network with m edges and integral capacities using an augmenting-paths algorithm.

(a) Show how to update the maximum flow in $O(m)$ time after increasing a specified capacity by 1.

(b) Show how to update the maximum flow in $O(m)$ time after decreasing a specified (positive) capacity by 1.

Problem 3. At lunchtime it is crucial for people to get to the food trucks as quickly as possible. We will show how to use maximum flow to solve this problem. Consider the following model. The building is represented by a graph $G = (V, E)$, where each room, landing, or other location is represented by a vertex and each corridor or stairway is represented by an edge. Each corridor has an associated capacity c, meaning that at most c people can pass through the corridor at once. Traversing a corridor from one end to the other takes one timestep. (Traversing a room takes zero time.)

(a) Suppose all people are initially in a single room s, and that the building has a single exit t. Show how to use maximum flow to find a fastest way to get everyone out of the building. Hint: create an auxiliary graph G' which has vertices to represent each room *at each time step*.

(b) Show that the same technique can be used when people are initially in multiple locations and there are multiple exits.

(c) Generalize the approach to where different corridors have different (integral) transit times.

OPTIONAL (d) Suppose transit times are not integral. Is there still a way to solve the problem?

OPTIONAL (e) The above algorithm is polynomial in the number of people. Can you improve it to be polynomial (in the graph size) regardless of the number of people?