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On the power of shared object types to implement one-resilient Consensus

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Summary. In this paper we study the ability of shared object types to implement Consensus in asynchronous shared-memory systems where at most one process may crash. More specifically, we consider the following question: Let $n \geq 3$ and $\mathcal{S}$ be a set of object types that can be used to solve one-resilient Consensus among n processes. Can $\mathcal{S}$ always be used to solve one-resilient Consensus among $n-1$ processes? We prove that for $n = 3$ the answer is negative, even if $\mathcal{S}$ consists only of *deterministic* types. (This strengthens an earlier result by the first author proving the same fact for *nondeterministic* types.) We also prove that, in contrast, for $n > 3$ the answer to the above question is affirmative.

Key words: Distributed algorithms – Fault-tolerance – Shared objects – Consensus

1 Background and overview

In this paper we consider some questions concerning fault-tolerant implementations of Consensus in asynchronous shared-memory systems. In such systems, some number of processes communicate with each other by accessing shared typed objects. Processes take steps in a completely asynchronous manner. In one step, a process may invoke an operation on a shared object. This causes the object to atomically change its state and return a response to the process invoking the operation. The new state entered by the object and the response returned to the operation are determined by the specification of the type to which the object belongs. A process may *crash* – i.e., stop taking steps before completing its execution. A process that does not crash is called *correct*.

In this model of computation, an algorithm involving n processes is *t-resilient*, where $t < n$, if each correct process completes the task it initiates, as long as no more than t processes crash. The algorithm is *wait-free* if it is $(n-1)$-resilient. Thus, in a wait-free algorithm, each correct process completes its task after a finite number of its own steps regardless of the progress made by other processes. In particular, it must do so, even if all other processes crash.

Consensus is a fundamental problem in fault-tolerant distributed computing because it captures the essence of many practical problems that require some form of agreement. In the Consensus problem, each process has a private input value and eventually may decide (irrevocably) on some value. An algorithm $\mathbf{A}$ for n processes *implements* (or *solves*) t-resilient Consensus among $n > t$ processes if every execution of $\mathbf{A}$ satisfies the following properties: (i) no two processes decide different values (Agreement); (ii) the value decided by any process is the input value of some process (Validity); and (iii) if no more than t processes crash in the execution then, eventually, each correct process must decide (Termination).

A set of types $\mathcal{S}$ *implements* (or *solves*) t-resilient Consensus among n processes, if there is an algorithm $\mathbf{A}$ that implements t-resilient Consensus among n processes and every object used by $\mathbf{A}$ belongs to a type in $\mathcal{S}$. Throughout this paper we assume that $\mathcal{S}$ contains type register. Objects of this type are shared registers that support (only) read and write operations. We refer to such objects simply as "registers".[1]

Let $\mathcal{S}$ be a set of types that implements t-resilient Consensus among n processes, for $n-1 > t$. It is easy to see that $\mathcal{S}$ can be used to solve t-resilient Consensus among $n+1$ processes – and thus any number of processes greater than n: n of

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[1] Several kinds of shared registers have been studied in the literature, differing in the number of processes that may read and write them, the number of values they can store, and the assurances they provide to concurrent read and write operations. It has been established that the strongest kind of registers – multi-reader, mutli-writer, multi-valued, atomic registers – can be implemented from the weakest kind – single-reader, single-writer, two-valued, safe registers (cf. [8] for more details and pointers to the relevant literature). Our formal model of shared types (cf. Sect. 2.1) commits us to multi-reader, multi-writer, atomic registers. In view of the above-mentioned result, however, this causes no loss of generality.

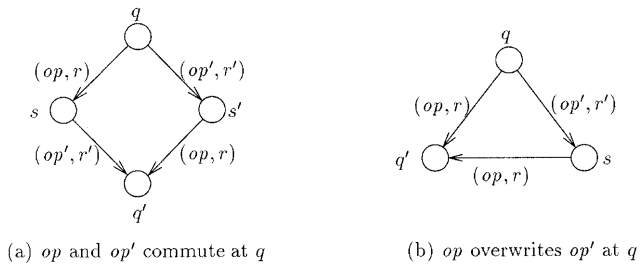
the processes solve Consensus using $\mathcal{S}$ and write the decision to a register; the remaining process obtains the decision from that variable. It is not nearly as clear whether $\mathcal{S}$ can also be used to solve t-resilient Consensus among $n-1$ processes – and thus any number of processes in the range $t+1$ and $n-1$. Now, reaching Consensus appears more difficult, since the algorithm can rely on fewer correct processes ($n-t-1$ instead of $n-t$). Thus, *prima facie*, it appears possible that a set of types is strong enough to solve t-resilient Consensus among n processes, but too weak to solve t-resilient Consensus among $n-1$ processes.

The question of whether this is possible was studied by Chandra et al. [4]. They showed that for any $n-1 > t \geq 2$, any set of types strong enough to solve t-resilient Consensus among n processes, is also strong enough to solve t-resilient Consensus among $n-1$ processes. The case of $t=1$ was left open in that investigation. This special case is an important one since, in practice, handling a single failure is often an adequate degree of fault-tolerance [14]. Subsequent to [4], the first author considered this special case and proved that the restriction to $t \geq 2$ is necessary. Specifically, in [10] he exhibited a *nondeterministic* type wor with the following property: Using only wor objects and registers, it is possible to implement one-resilient Consensus for three, but not for two, processes.

In this paper we complete the study of the special case of $t=1$. First, we strengthen the result of [10] by exhibiting a *deterministic* type that can be used (together with registers) to implement one-resilient Consensus for three, but not two, processes. This object type has another interesting property: To our knowledge it is the only known *deterministic* type at level one of the Consensus hierarchy [6] which cannot be implemented in a one-resilient manner for three or more processes using only registers.[2] Second, and in contrast, we show that for any $n \geq 4$, a set of (deterministic or nondeterministic) types strong enough to implement one-resilient Consensus among n processes is also strong enough to implement one-resilient Consensus among $n-1$ processes. The proof of this result is based on a variant of techniques previously employed by Borowsky and Gafni [1] (see also [13,2]) and Chandra et al. [4].

The rest of the paper is organised as follows: In Sect. 2, we define formally the model of computation. In Sect. 3 we define the deterministic type that can be used to implement one-resilient Consensus for three, but not two, processes. In Sect. 4 we prove that for all $n \geq 4$, a set of types strong enough to implement one-resilient Consensus among n processes, is also strong enough to implement one-resilient Consensus among $n-1$ processes. We conclude in Sect. 5.

[2] The level of a type in the Consensus hierarchy is the maximum positive integer n such that we can implement wait-free (i.e., $(n-1)$-resilient) Consensus among n processes using only objects of that type and registers. (If there is no such maximum, the level of the type in the Consensus hierarchy is ∞.) The significance of this hierarchy lies in the following fundamental result, due to Herlihy [6]: If a type T is at level n of the Consensus hierarchy, then it is possible to use only objects of type T and registers to give a wait-free implementation of *any* object type in a system of n processes.



(a) op and op' commute at q (b) op overwrites op' at q

Fig. 1. Commutative and overwriting operations

2 The model of computation

2.1 Types and objects

An *object type* T is a tuple (ST, OP, RES, δ). ST is a set of *states*, OP is a set of *operations*, RES is a set of *responses* to operations, and $\delta : ST \times OP \rightarrow 2^{RES \times ST}$ is a *state transition function*. The state transition function of T describes the behaviour of an object of type T, and is sometimes called the type's *sequential specification*. Informally, $(res, q') \in \delta(q, op)$ means that if the current state of the object is q and operation op is applied to it, then it is possible that res will be returned to the operation and the object will enter state q'. For convenience, we require that the object's response to an operation applied at some state uniquely determines the object's new state. More precisely, for any $(q, op) \in ST \times OP$, if $\delta(q, op)$ contains (res, q') and (res, q''), then $q' = q''$.

T *exhibits finite nondeterminism* if $\delta(q, op)$ is finite, for all $(q, op) \in ST \times OP$. T is *deterministic* if $|\delta(q, op)| \leq 1$, for all $(q, op) \in ST \times OP$. T is *total* if $\delta(q, op) \neq \emptyset$, for all $(q, op) \in ST \times OP$. If T is deterministic and total, $\delta(q, op)$ contains exactly one element. In this case, we will slightly abuse notation and write $\delta(q, op)$ to denote that element (rather than the set that contains it). In other words, for deterministic total types we will view δ as a function from $ST \times OP$ to $RES \times ST$ (rather than to $2^{RES \times ST}$).

Let op and op' be operations of a deterministic total type T, q be a state of T, and δ be the state transition function of T. The two operations *commute* at q if applying them in either order has the same effect. More precisely, there exist responses r, r' and states s, s', q' such that $\delta(q, op) = (r, s)$, $\delta(s, op') = (r', q')$, $\delta(q, op') = (r', s')$, and $\delta(s', op) = (r, q')$; see Fig. 1(a). Operation op *overwrites* op' at q, if applying op after op' at q has the same effect as applying op by itself. More precisely, there exist responses r, r' and states s, q' such that $\delta(q, op') = (r', s)$ and $\delta(q, op) = \delta(s, op) = (r, q')$; see Fig. 1(b).

An *object* is an instance of an object type. For the purposes of this paper, we can think of an object O of type T as an automaton whose states and state transition function are as in T – except that states are labeled with O, to distinguish them from the states of other objects of the same type.

2.2 Processes and algorithms

A *process* is a deterministic automaton that interacts with objects. More precisely, let $\mathcal{A}$ be a set of objects, and OP, RES be the set of all operations and responses, respectively, of the types to which the objects in $\mathcal{A}$ belong. We define a *process*

that uses $\mathcal{A}$ as a tuple $P = (\Sigma, \Sigma_0, \nu, \tau)$, where Σ is a set of *states*; $\Sigma_0 \subseteq \Sigma$ is a set of *initial states*; and the functions $\nu : \Sigma \to OP \times \mathcal{A}$, and $\tau : \Sigma \times RES \to \Sigma$ describe the interaction of the process with the objects. Intuitively, if P is in a state $\sigma \in \Sigma$ and $\nu(\sigma) = \langle op, O\rangle$, then in its next step P will apply operation op to object O. Based on its own current state, O will return a response res to P and will enter a new state, in accordance with the state transition function of the type to which O belongs. Finally, P will enter state $\tau(\sigma, res)$, as a result of the response it received from O.

An *algorithm* $\mathbf{A}$ consists of a set of processes Π, a set of objects $\mathcal{A}$ so that each $P \in \Pi$ uses a subset of $\mathcal{A}$, and an initial state for each object in $\mathcal{A}$. The designated initial state of an object is one of the states of the type to which the object belongs. If $\mathcal{A}$ is finite, then $\mathbf{A}$ is a *bounded* algorithm. A *configuration* C of $\mathbf{A}$ is a tuple consisting of the state of each process in Π and each object in $\mathcal{A}$. C is an *initial* configuration of $\mathbf{A}$ if each process is in one of its initial states and each object is in the state designated as the initial state by $\mathbf{A}$.

A *step* of process P is a tuple (P, op, O, res); this indicates that P has applied operation op to object O and received response res. Let $P = (\Sigma, \Sigma_0, \nu, \tau)$ and C be a configuration, where the state of P in C is σ. If $\nu(\sigma) = \langle op, O\rangle$, we say that P has operation op to object O *pending* in C. If, in addition, the state of O in C is q, and $(res, q') \in \delta(q, op)$ (where δ is the state transition function of O's type), then we say that the step $e = (P, op, O, res)$ is *applicable* to C. If $e = (P, op, O, res)$ is applicable to C, $e(C)$ denotes the configuration resulting from C after step e. More precisely, if in C the state of P is σ and the state of O is q, then $e(C)$ is the configuration in which all processes other than P and all objects other than O are in the same state as in C, P has state $\tau(\sigma, res)$, and O is in state q' such that $(res, q') \in \delta(q, op)$, where δ the state transition function of O's type. (Note that q' is well-defined by our requirement that object O's response to operation op applied in state q uniquely determines O's new state, q'.)

A *schedule* S *of algorithm* $\mathbf{A}$ is a (finite or infinite) sequence of steps of $\mathbf{A}$'s processes. If every step in S is a step of process P, S is a *solo* schedule of P. We say that $S = e_1, e_2, \ldots, e_i, \ldots$ is applicable to a configuration C, if e_1 is applicable to C, and e_{i+1} is applicable to $e_i(e_{i-1}(\ldots(e_1(C))\ldots))$, for all i. If S is finite and has k steps, $S(C)$ denotes $e_k(e_{k-1}(\ldots(e_1(C))\ldots))$, i.e., the configuration that results after applying the steps in S one at a time, starting with configuration C. If S and S' are schedules, $S \cdot S'$ denotes their concatenation. For any configurations C and C', we say that C' is *reachable* from C, if there is a schedule S applicable to C such that $C' = S(C)$.

Let S be any infinite schedule applicable to an initial configuration I of algorithm $\mathbf{A}$. We say that a process P *crashes* (or is *faulty*) in S if P has only finitely many steps in S; otherwise, we say that P is *correct* in S. The idea of modeling correct processes as ones that take infinitely many steps in an infinite execution – and crashed processes as ones that take only finitely many steps – was introduced in [5]. It turns out to be a very convenient convention. We can easily accommodate processes that terminate in this framework by imagining that a process that reaches a final state takes infinitely many "do-nothing" steps thereafter.

An *execution* of algorithm $\mathbf{A}$ is a pair (I, S), where I is an initial configuration of $\mathbf{A}$ and S is an infinite schedule of $\mathbf{A}$ applicable to I.

2.3 The Consensus problem

In the Consensus problem, each process has a private initial value, drawn from the set $\{0, 1\}$. Thus, each initial configuration of a Consensus algorithm may be associated with a function mapping each of the processes to $\{0, 1\}$. Some states of each process are associated with an irrevocable decision in $\{0, 1\}$. The decision is irrevocable in the sense that if a process enters a state associated with decision d, then all states it may subsequently enter are also associated with d. Let n, t be integers such that $n > t \geq 1$. A *t-resilient Consensus algorithm* $\mathbf{A}$ for n processes satisfies the following properties. For any initial configuration I of $\mathbf{A}$, and any schedule S of $\mathbf{A}$ applicable to I,

Agreement: If two processes have decided in $S(I)$, then their decisions are the same.

Validity: If a process has decided in $S(I)$, then its decision must be the initial value of some process in I.

Termination: If S is infinite and at most t processes crash in S, then for any correct process P in S, there is a prefix S' of S such that P has decided in $S'(I)$.

If $t = n-1$, we say that $\mathbf{A}$ is a *wait-free* Consensus algorithm for n processes.

A set of types $\mathcal{S}$ *implements* t-resilient (wait-free) Consensus for n processes if there is a t-resilient (wait-free) Consensus algorithm $\mathbf{A}$ for n processes. If, in addition, $\mathbf{A}$ is bounded, then we say that $\mathcal{S}$ *boundedly implements* t-resilient (wait-free) Consensus for n processes.

3 The (exceptional) case of three processes

In this section we introduce a deterministic type, named dor[3], that (together with registers) is strong enough to solve one-resilient Consensus for three processes, but not for two processes. The type is defined in Sect. 3.1. We prove that it is strong enough to solve one-resilient Consensus for three processes in Sect. 3.2. Finally, we prove that it is too weak to solve one-resilient Consensus for two processes (equivalently: wait-free Consensus for two processes) in Sect. 3.3.

3.1 Specification of dor

The specification of dor is motivated by a specific algorithm that we have in mind for implementing one-resilient Consensus among three processes using an object of this type and registers. Thus, it is best to explain dor alongside an informal description of that algorithm.

A process can apply two kinds of operations to a dor object: "enroll" an integer in the set $\{0, \ldots, 5\}$, and "query"

[3] dor stands for deterministic one-resilient.

whether 0 was enrolled before 1. In the algorithm that implements one-resilient Consensus among three processes Q_0, Q_1 and Q_2, each process enrolls two integers to a dor object O: Q_0 enrolls 0 and 3, Q_1 enrolls 1 and 4, and Q_2 enrolls 2 and 5. The three processes follow a particular pattern in enrolling their integers. Q_0 and Q_1 follow the same pattern, while Q_2 follows a slightly different one. Specifically, each $Q \in \{Q_0, Q_1\}$ enrolls its first integer and waits for one of the other two processes to do the same. Q then enrolls its second integer and, again, waits for one of the other two processes to do the same. In contrast, Q_2 *first* waits for one of the other two processes to enroll its first integer and then enrolls its own first integer. Similarly, Q_2 waits for one of the other two processes to enroll its second integer and then enrolls its own second integer. Thus, Q_0 and Q_1 follow the pattern: enroll, wait, enroll, wait; while Q_2 follows the pattern: wait, enroll, wait, enroll. Note that, as a result of this protocol, the first integer to be enrolled is either the first integer enrolled by Q_0 or the first integer enrolled by Q_1 – i.e., 0 or 1.

After enrolling its integers in this manner, each process Q_k "queries" O to determine whether 0 was enrolled before 1. This allows the three processes to agree on which of the two integers was enrolled first – and thus to agree on the identity of one of Q_0 or Q_1, so that if they agree on Q_k, then Q_k has taken at least one step (the step required to enroll k). By requiring each of Q_0 and Q_1 to write its input value into a shared register before enrolling its first integer, we can easily turn agreement on the identity of one of these two processes into agreement on that process' input value. In this way, we can solve one-resilient Consensus among the three processes using only one dor object and some registers.

We must make special provisions to ensure that dor cannot be used to implement one-resilient Consensus for *two* processes. Informally, we shall accomplish this by ensuring that any use of a dor object that does not conform to the one described above puts the object in a state from which it is impossible to get "useful" information. In particular, it is impossible to reliably determine whether 0 or 1 was the "winner" (i.e., was enrolled first). The nondeterministic type wor described in [10] did this by defining an "upset" state that a wor object enters if not accessed as required by the algorithm described above. The object responds by nondeterministically returning 0 or 1 if queried about the winner while in that state. We cannot use this technique here because we want to define a *deterministic* type. Instead, we design the type to return deterministic, but misleading, responses if accessed "inappropriately". Doing so while ensuring that the resulting type is not strong enough to implement one-resilient Consensus for two processes turns out to be nontrivial.

We now describe the type dor in more detail. It supports the following operations: $\text{ENROLL}(v)$, for $v \in \{0, 1, \ldots, 5\}$, and REVEAL. $\text{ENROLL}(v)$ is used to enroll v. It may change the state of the object but always returns the same response, ack. A REVEAL is used to find out if 0 was enrolled before 1. It does not change the state, and returns a value, 0 or 1. Intuitively, response 1 is supposed to indicate that 1 was enrolled before 0 (or that neither has been enrolled yet); while response 0 is supposed to indicate that 0 was enrolled before 1. Under certain conditions, however, even though 0 was enrolled before 1, the object will (misleadingly) respond 1 to such a question. This will be the case if the question is asked "prematurely", or after the object has been "mishandled". The question is premature if it is asked before at least two of $\{3, 4, 5\}$ have been enrolled. The object is mishandled if and only if, starting with the enrollment of 0, one of the following conditions takes place: (a) any number other than 5 is enrolled more than once, or (b) 5 is enrolled more than once after both 2 and one of $\{3, 4\}$ have been enrolled.

Each node of the graph below represents a state of dor. Let δ denote the state transition function of type dor, and q be any node of the graph shown below.

$\text{ENROLL}(v)$ for $v \in \{0, \ldots, 5\}$: If there is an arc labeled with v from node q to node q', then $\delta(q, \text{ENROLL}(v)) = (ack, q')$; otherwise, $\delta(q, \text{ENROLL}(v)) = (ack, ☹)$.

REVEAL: If q is a black node then $\delta(q, \text{REVEAL}) = (0, q)$; otherwise ($q$ is a white node), $\delta(q, \text{REVEAL}) = (1, q)$.

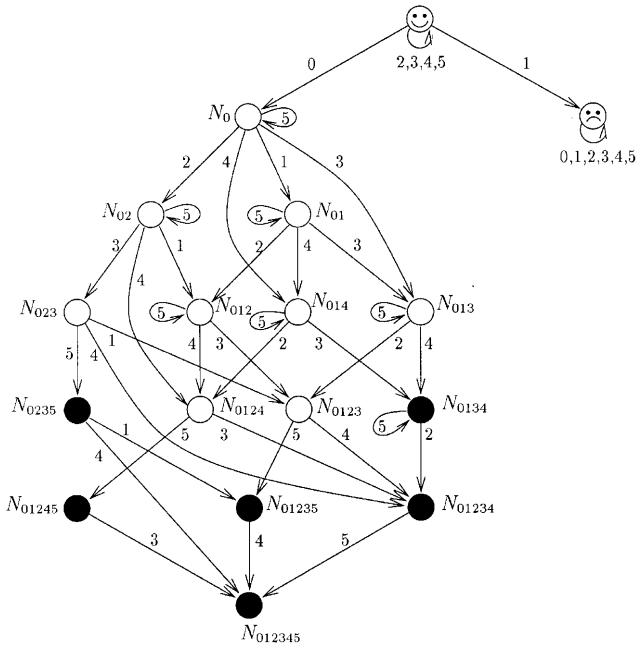


Fig. 2. State transition function of type dor

The behaviour of dor is formally specified by the state transition diagram in Fig. 2. The type has seventeen states, represented as nodes in the diagram. The state labeled ☺ is called the *fresh* state; the state labeled ☹ is called the *upset* state; the remaining are called *normal* states. The colour of a node (black or white) indicates the response (0 or 1, respectively) of a REVEAL applied to the corresponding state. Recall that each ENROLL operation returns ack, so there is no need to explicitly represent that in the diagram.

The directed edges of the diagram represent transitions caused by ENROLL operations (recall that REVEAL operations do not affect the state). The label of an edge denotes the parameter of the corresponding ENROLL. For example, an $\text{ENROLL}(0)$ operation applied to the fresh state causes dor to enter the normal state N_0. Note that, in each state other than fresh, one or more edges (corresponding to one or more ENROLL operations) are missing. In that case, there is an implicit transition to the upset state. Thus, for instance, node N_{01} has implicit transitions labeled 0 and 1 to the upset state. In terms of our earlier informal description of dor, implicit

transitions to the upset state correspond to operation invocations that "mishandle" the type. Thus, after such a transition has taken place, the type will (misleadingly) respond 1 to a REVEAL, even though 0 was enrolled before 1.

The fresh state is a source, in the sense that it is not reachable from any other state. The upset state is a sink, in the sense that no other state is reachable from it. Intuitively, it is the state reached if the object is initialised to the fresh state and either 1 is enrolled before 0, or the object is mishandled. The normal states are the states reached if the object is initialised to the fresh state, 0 is enrolled before 1, and the object is not mishandled. In the diagram, each normal state is denoted by a node labeled $N_{v_1 \ldots v_i}$, where $\{v_1, \ldots, v_i\}$ is a nonempty subset of $\{0, 1, \ldots, 5\}$. Intuitively, the subscript of a normal state represents the set of numbers enrolled "so far", assuming the object is initialised in the fresh state. However, some enrollments are "ineffectual" (they do not affect the state), and some enrollments are "implicit" (they occur as a side-effect of other enrollments). Specifically, any attempt to enroll a number before enrolling 0 or 1 is ineffectual. (This corresponds to the self-loop at the fresh state.) Also, any attempt to enroll 5 before both 2 and one of $\{3, 4\}$ have been enrolled is ineffectual. (This corresponds to the self-loops in the normal states.) An implicit enrollment of 1 is forced if an attempt is made to enroll 4 before 1 has been enrolled, or to enroll 3 before both 1 and 2 have been enrolled. (This corresponds to the state transitions that "jump" a level in the diagram – e.g., the transitions labeled 3 and 4 at N_0.) Note that the subscripts of the white normal states are precisely those that contain at most one of $\{3, 4, 5\}$. In other words, they are precisely the states where at most one of $\{3, 4, 5\}$ has been (effectively) enrolled. Recall, from our earlier informal discussion of the type, that these are the states in which REVEAL returns a misleading response because the question whether 0 was enrolled before 1 is asked "prematurely".

We use the following terminology for the state transitions of type dor. Let op be an operation, q be a state and δ be the state transition function of dor. We say that op is *stationary* at q if $\delta(q, op) = (r, q)$, for some response r. In terms of Fig. 2, this corresponds to the self-loop transitions. We say that op is *upsetting* at q if $\delta(q, op) = (r, ☹)$, for some response r. This corresponds to (implicit and explicit) transitions leading to the upset state in the diagram. We say that op is *ordinary* at q, if op is neither stationary nor upsetting at q. This corresponds to explicitly represented transitions on the left side of the diagram (i.e., all edges explicitly shown, except for the self-loops at the fresh and upset states, and the edge from the fresh to the upset state).

The reader should observe that, with very few exceptions, in each state of the type dor, any two operations either commute, or one overwrites the other. The specification of dor was crafted deliberately in this manner. As we shall see in Sect. 3.3, this turns out to be crucial in proving that dor cannot implement one-resilient Consensus between two processes.

3.2 Type dor *solves one-resilient Consensus for three processes*

In this subsection we prove that dor (together with registers) implements one-resilient Consensus for three processes. An algorithm that does this was described informally in the previous subsection; it is shown in pseudocode in Fig. 3. In the algorithm, and throughout the paper, if $k \in \{0, 1\}$, we denote by $\overline{k}$ the complement of k, i.e., $\overline{k} = 1 - k$. The notation Apply(Q, op, O) denotes the procedure by which process Q applies operation op to object O; the procedure returns the response of the object to that operation. Register R_k, $k \in \{0, 1, 2\}$, is used by process Q_k to indicate how many numbers it has enrolled so far. Before enrolling their first numbers, Q_0 and Q_1 write their input values into registers D_0 and D_1, respectively.

We now turn to the correctness proof of the algorithm.

Shared O: dor, initialised to state ☺
D_0, D_1: register, each initialised to $\perp$
R_0, R_1, R_2: register, each initialised to 0

Code for process Q_k, for $k \in \{0, 1\}$
1 D_k := initial value of Q_k
2 **for** ℓ := 1, 2 **do**
3 Apply$(Q_k, \text{ENROLL}(3(\ell - 1) + k), O)$
4 $R_k := \ell$
5 **wait until** $R_{\overline{k}} \geq \ell$ **or** $R_2 \geq \ell$
6 i := Apply(Q_k, REVEAL, O)
7 **decide** D_i

Code for process Q_2
1 **for** ℓ := 1, 2 **do**
2 **wait until** $R_0 \geq \ell$ **or** $R_1 \geq \ell$
3 Apply$(Q_2, \text{ENROLL}(3(\ell - 1) + 2), O)$
4 $R_2 := \ell$
5 i := Apply(Q_2, REVEAL, O)
6 **decide** D_i

Fig. 3. A one-resilient Consensus algorithm for three processes using objects of type dor and registers

Lemma 1 *Consider any execution of the algorithm in Fig. 3, in which at most one process crashes. In this execution, no correct process is stuck forever in the* **wait** *statement of its code (line 5 of Q_0 and Q_1, or line 2 of Q_2).*

Proof. Suppose, for contradiction, that a correct process Q_k, for some $k \in \{0, 1, 2\}$, is stuck in the **wait** statement in the execution. There are two cases:

Case 1. Q_k is stuck in the first iteration of the **for** *loop:* We first claim that $k \neq 2$. If not, then by line 2 of its code, Q_2 must always read $R_0 = 0$ and $R_1 = 0$ when it executes the **wait** statement during its first iteration of the **for** loop. This is possible only if both Q_0 and Q_1 crash, before executing line 4, during their first iteration of the **for** loop. This contradicts our hypothesis that at most one process may crash.

We next argue that $k \notin \{0, 1\}$. Suppose, for contradiction, that process Q_k is stuck in the **wait** statement in line 5, for some $k \in \{0, 1\}$. Then Q_k must always read $R_{\overline{k}} = 0$ and $R_2 = 0$ in executing that line. Since $R_{\overline{k}}$ is always equal to 0, it follows that $Q_{\overline{k}}$ crashes before executing its line 4 during its first iteration of the **for** loop. Similarly, as R_2 always equal to 0, it must be that Q_2 crashes before executing its line 4 during its first iteration of the **for** loop (because we have just shown that if Q_2 is correct then in the first iteration of the **for** loop Q_2 cannot be stuck forever in line 2 and hence it must

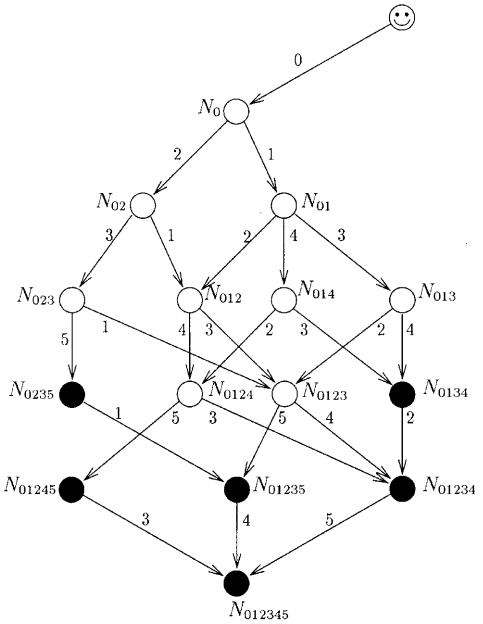


Fig. 4. Possible state transitions of object O of the algorithm in Fig. 3, in which the first operation applied to O is $\text{ENROLL}(0)$

eventually execute line 4). Thus, both $Q_{\overline{k}}$ and Q_2 are faulty. This again contradicts the assumption that there is at most one faulty process.

Case 2. Q_k is stuck in its second iteration of the **for** *loop:* We have just shown that no correct process can be stuck when executing its first iteration of the **for** loop. Hence, every correct process executes the second iteration of the **for** loop. Using this fact and an argument similar to the one in Case 1, we can prove that no correct process can be stuck in the **wait** statement during the second iteration of the **for** loop, as well.

Since each of these two cases leads to a contradiction, the lemma follows. □

Lemma 2 *Consider any execution of the algorithm in Fig. 3. Let d be the parameter of the first* ENROLL *operation applied to the* dor *object O in that execution. Then, $d \in \{0, 1\}$ and every* REVEAL *operation applied to O during that execution returns response d.*

Proof. By the specification of the algorithm, it can be readily verified that the first operation applied to the dor object O is either $\text{ENROLL}(0)$ or $\text{ENROLL}(1)$. Thus, $d \in \{0, 1\}$. We consider these two cases in turn.

Case 1. $d = 1$: We show that whenever a REVEAL operation is applied, object O is in state ☹. Since the REVEAL operation returns 1 at state ☹, it follows that every REVEAL operation applied to O returns 1. To this end, first note that $\text{ENROLL}(1)$ is upsetting at ☺. Since the initial state of O is ☺ and the first operation applied to O in the execution is $\text{ENROLL}(1)$, O enters state ☹ after the first operation is applied. Since every operation is stationary at ☹, O stays in state ☹ throughout the rest of the execution. Consequently, whenever a process applies REVEAL, object O is in state ☹ and returns 1, as required.

Case 2. $d = 0$: We need to show that every REVEAL operation applied to O gets back response 0. To do so, it suffices to show that a REVEAL operation will be applied only if O is in a black state, since the REVEAL operation returns 0 in all black states. By Fig. 3, in any execution of the algorithm:

(a) for each $k \in \{0, 1, 2\}$, the sequence of operations that process Q_k applies to O is (a prefix of) $\text{ENROLL}(k)$, $\text{ENROLL}(k+3)$, REVEAL;
(b) at least two of $\{\, \text{ENROLL}(v) : 0 \le v \le 2 \,\}$ are applied before both $\text{ENROLL}(3)$ and $\text{ENROLL}(4)$ (because before executing its second iteration of the **for** loop, each of Q_0 and Q_1 waits for another process to apply its first ENROLL operation);
(c) either $\text{ENROLL}(3)$ or $\text{ENROLL}(4)$ is applied before $\text{ENROLL}(5)$ (because before applying $\text{ENROLL}(5)$ to O, Q_2 waits for at least one of $\{Q_0, Q_1\}$ to apply its second ENROLL operation); and
(d) at least two of $\{\, \text{ENROLL}(v) : 3 \le v \le 5 \,\}$ are applied before a REVEAL (because before applying REVEAL to O, every process waits for another process to apply its second ENROLL operation).

Using (a)–(c), it is easy to verify that if the first operation applied to O in the execution is $\text{ENROLL}(0)$, then O will always be in a normal state – i.e., O never enters state ☹ – during the execution. To assist with this verification, in Fig. 4 we have shown all the possible state transitions of object O in response to the ENROLL operations applied under the algorithm in Fig. 3 when $\text{ENROLL}(0)$ is the first operation applied.

By (a) and (d), when a process applies a REVEAL operation to O, at least two of the three processes have each applied exactly two ENROLL operations. By inspection of Fig. 4, it is easy to see that O is in N_{0123}, N_{0124}, or a black state. (Note that N_{0123} and N_{0124} are white.) We claim that if O is in N_{0123} or N_{0124}, no process will apply REVEAL, until at least one more ENROLL operation is performed – which will put O in a black state. This is because in state N_{0123}, only Q_0 has done two ENROLL operations (with parameters 0 and 3), while each of Q_1 and Q_2 has done only one (with parameters 1 and 2, respectively). Thus, if O is in N_{0123}, by (a), the only process whose next access to O is a REVEAL is Q_0. By (d), however, Q_0 must wait until Q_1 or Q_2 has done its second ENROLL operation before it can apply REVEAL to O. A similar argument (interchanging the roles of Q_0 and Q_1) applies for state N_{0124}.

This shows that before *any* process applies REVEAL, O must have entered a black state. Furthermore, it can be immediately seen from Fig. 4 that once O enters a black state, it remains in a black state throughout the execution. Thus, whenever a process applies a REVEAL, O is in a black state and thus returns 0 to the REVEAL operation, as wanted. □

Theorem 1 *The algorithm in Fig. 3 is a one-resilient Consensus algorithm for three processes that uses only objects of type* dor *and registers.*

Proof. It is clear that the algorithm in Fig. 3 uses only objects of type dor and registers. For the rest of the proof, fix an arbitrary execution of the algorithm in which at most one process crashes. We need to show that this execution satisfies the three properties of Consensus – Termination, Validity and Agreement.

Termination: By Lemma 1, no correct process can be stuck in any execution of the algorithm. Therefore, every correct process decides in the execution.

Validity: By Fig. 3, it is clear that, in any execution of the algorithm, only process Q_k writes into D_k, and the value Q_k writes there is its initial value, for each $k \in \{0, 1\}$. Therefore, to show that the execution satisfies Validity, it suffices to show that if a process decides the value in D_d, for some $d \in \{0, 1\}$, then Q_d has previously written into D_d. This follows from Lemma 2, since in every execution of the algorithm, a process writes into D_k before applying $\text{ENROLL}(k)$ to O, for each $k \in \{0, 1\}$.

Agreement: Suppose that two distinct processes Q and Q' have decided in the execution (otherwise, Agreement is trivially satisfied). By Lemma 2 and the specification of the algorithm (lines 6–7 of Q_0 and Q_1, and lines 5–6 of Q_2), both Q and Q' decide the value in the same register D_d, where $d \in \{0, 1\}$ is the parameter of the first ENROLL operation applied to the dor object O during the execution. As just argued, a process reads D_d only after Q_d has written into it. Since only one value is written into D_d, Q and Q' decide the same value. Thus Agreement is satisfied. □

3.3 Type dor *does not solve wait-free Consensus for two processes*

In this subsection we show that, using only objects of type dor and registers, we cannot solve wait-free Consensus for two processes. (Recall that the concepts of wait-freedom and one-resilience coincide for two-process algorithms.) The argument uses a technique introduced in [10], and is outlined in the following three paragraphs.

Assume, by way of contradiction, that there exists a wait-free Consensus algorithm $\mathbf{A}$ for two processes P_0 and P_1 that uses only objects of type dor and registers. Assume, further, that $\mathbf{A}$ uses a minimal number of dor objects. Since wait-free Consensus for two processes is not solvable using only registers [12], it follows that $\mathbf{A}$ uses at least one dor object.

To derive a contradiction, we proceed in two stages. In the first stage, we show that the assumed algorithm $\mathbf{A}$ must have a special configuration, $\mathcal{B}_0$, and a dor object, $\mathcal{O}$, with the following properties: For each $k \in \{0, 1\}$, process P_k has an $\text{ENROLL}(k + 2)$ operation to $\mathcal{O}$ pending in $\mathcal{B}_0$. Starting from $\mathcal{B}_0$, after applying $\text{ENROLL}(k + 2)$ to $\mathcal{O}$, P_k cannot apply $\text{ENROLL}(v)$ to $\mathcal{O}$, for any $v \in \{0, 1, 2, 3\}$, and cannot decide unless it applies

- an $\text{ENROLL}(4)$ to $\mathcal{O}$; or
- an $\text{ENROLL}(5)$ to $\mathcal{O}$, after the other process has taken at least one step.

Furthermore, if each process is about to apply to $\mathcal{O}$ the $\text{ENROLL}(4)$ or $\text{ENROLL}(5)$ that it must execute before deciding, one process must be about to apply $\text{ENROLL}(4)$ while the other must be about to apply $\text{ENROLL}(5)$: they cannot both be about to apply ENROLL with the same parameter.

In the second stage of the proof, starting from configuration $\mathcal{B}_0$, we show how to solve Consensus for two processes without using the dor object $\mathcal{O}$. Roughly speaking, the fact that, starting from $\mathcal{B}_0$, the two processes must apply ENROLL operations to $\mathcal{O}$ with *different* parameters is exploited to satisfy Agreement, while the fact that the process that is to apply $\text{ENROLL}(5)$ must do so in a nonsolo execution is exploited to satisfy Validity. In this way we construct a Consensus algorithm for two processes that uses one dor object less than $\mathbf{A}$. This contradicts our hypothesis that the number of dor objects used in $\mathbf{A}$ is minimal. This contradiction means that the assumed wait-free Consensus algorithm $\mathbf{A}$ that uses only dor objects and registers does not exist.

3.3.1 Preliminaries

We first review the definition of valence for configurations of a Consensus algorithm, which was first defined in [5]. Let C be any configuration of $\mathbf{A}$. We say that C is *bivalent* if there are configurations C_0 and C_1, reachable from C, such that some process decides 0 in C_0 and some process decides 1 in C_1. C is *v-valent*, for $v \in \{0, 1\}$, if there is no configuration C' reachable from C so that some process decides $\overline{v}$ in C'. C is *univalent* if it is either 0-valent or 1-valent. We say that two univalent configurations *have the same valence* if they are both 0-valent or both 1-valent, and *have opposite valence* if one is 0-valent and the other is 1-valent. By the definitions of univalent and bivalent configuration it follows that if a configuration C of $\mathbf{A}$ is neither univalent nor bivalent, then no process decides in any configuration reachable from C. Since this contradicts the Termination property, we conclude that every configuration of a Consensus algorithm is either univalent or bivalent.

We establish some elementary properties of configurations that will be used subsequently.

Lemma 3 *Let C be any configuration of $\mathbf{A}$, and P be any process. There exists a solo schedule S of process P such that S is applicable to C and P has decided in $S(C)$.*

Proof. Immediate from the Termination requirement of Consensus and the fact that algorithm $\mathbf{A}$ is wait-free. □

Lemma 4 *Let C and C' be any two univalent configurations such that the state of some process and each object is the same in C as in C'. Then, C and C' have the same valence.*

Proof. Let P be the process whose state is the same in C as in C', and S be a solo schedule of P applicable to C such that P has decided in $S(C)$. Because the state of P and each object is the same in C as in C', by induction on the length of S, we can show that S is also applicable to C'. This implies that P has decided the same value, say $v \in \{0, 1\}$, in both $S(C)$ and $S(C')$. Thus, both $S(C)$ and $S(C')$ are v-valent. Since C and C' are univalent, it follows that they must also be v-valent. □

Lemma 5 *Let C and C' be any configurations of $\mathbf{A}$. Let S be any finite schedule applicable to C such that (i) every process that takes a step in S and every object, except one* dor *object O, is in the same state in C as in C'; (ii) O is in a white state in C' and in $S'(C)$, for every prefix S' of S; and (iii) every operation applied to O in S is stationary at the state of O in C'. Then,*

- *S is applicable to C',*

- *every process that takes a step in S and every object except O is in the same state in $S(C)$ as in $S(C')$, and*
- *O is in the same state in C' as in $S(C')$.*

Proof. Let S^i be the prefix of S of length i. Recall that a REVEAL operation applied to a dor object returns 1 if the object is in a white state, and that a stationary operation applied to a dor object does not change the object's state. Using these two properties, by a straightforward induction on i, we can show that

> S^i is applicable to C', every process that takes a step in S and every object except O is in the same state in $S^i(C)$ as in $S^i(C')$, and O is in the same state in C' as in $S^i(C')$.

The lemma follows immediately from this statement. □

3.3.2 Stage one of the proof

We now show the existence of a configuration and a dor object of $\mathbf{A}$ that satisfy the properties described at the beginning of Sect. 3.3.

Lemma 6 *There exist a bivalent configuration $\mathcal{C}$ of $\mathbf{A}$, steps e_0 and e_1 of processes P_0 and P_1, respectively, applicable to $\mathcal{C}$, and an object $\mathcal{O}$ such that*

- *$e_0(\mathcal{C})$ and $e_1(\mathcal{C})$ are univalent and have opposite valence, and*
- *e_0 and e_1 both access $\mathcal{O}$.*

Proof. Using a, by now, standard argument [5,6], it can be shown that $\mathbf{A}$ has a bivalent initial configuration I. From this, and the fact that $\mathbf{A}$ is wait-free, it follows that there is a bivalent configuration $\mathcal{C}$, reachable from I, so that any step applicable to $\mathcal{C}$ leads to a univalent configuration. Since $\mathbf{A}$ uses deterministic types, for each $k \in \{0,1\}$, P_k has exactly one step, e_k, applicable to $\mathcal{C}$. Since $\mathcal{C}$ is bivalent while $e_0(\mathcal{C})$ and $e_1(\mathcal{C})$ are univalent, it follows that $e_0(\mathcal{C})$ and $e_1(\mathcal{C})$ have opposite valence.

It remains to prove that e_0 and e_1 access the same object. If not, it is easy to see that $e_0(e_1(\mathcal{C})) = e_1(e_0(\mathcal{C}))$, which contradicts that $e_0(\mathcal{C})$ and $e_1(\mathcal{C})$ have opposite valence. □

In the rest of this section we focus exclusively on configurations that are reachable from the bivalent configuration $\mathcal{C}$. Without loss of generality, we assume that $e_0(\mathcal{C})$ is 0-valent and $e_1(\mathcal{C})$ is 1-valent. (If not, we simply interchange the roles of e_0 and e_1 throughout the argument presented below.) In what follows, let q^* denote the state of $\mathcal{O}$ in configuration $\mathcal{C}$.

Since this stage of the proof is rather long, we give a roadmap to help explain its overall structure. We first prove that $\mathcal{O}$ is a dor object (Lemma 8) and that q^* is the fresh state (Lemma 10). The proof of this makes heavy use of the earlier-noted fact that dor exhibits a great deal of commutativity and overwriting between operations. We then prove that there is a solo schedule E_0 of P_0 so that P_0 has ENROLL(2) to $\mathcal{O}$ pending in $E_0(e_1(e_0(\mathcal{C})))$ (Lemma 11). In addition, there is a solo schedule E_1 of P_1 so that P_1 has ENROLL(3) to $\mathcal{O}$ pending in $E_1(E_0(e_1(e_0(\mathcal{C}))))$ (Lemma 12). Finally, we show that configuration $\mathcal{B}_0 = E_1(E_0(e_1(e_0(\mathcal{C}))))$ and dor object $\mathcal{O}$ (defined above in Lemma 6) have the properties stated at the beginning of Sect. 3.3 (Lemmata 15 and 16).

Lemma 7 *The two operations pending in $\mathcal{C}$ do not commute at q^*, and neither one of them overwrites the other at q^*.*

Proof. Using a standard argument [6,12] we show that the operations pending in $\mathcal{C}$ neither commute nor overwrite one another. Let op_0 and op_1 denote the operations to $\mathcal{O}$ of P_0 and P_1, respectively, that are pending in $\mathcal{C}$. Let δ be the state transition function of $\mathcal{O}$'s type.

Operations op_0 and op_1 do not commute at q^*: If not, then by definition there exist responses r_0, r_1 and states q_0, q_1, q such that, for each $k \in \{0,1\}$, $\delta(q^*, op_k) = (r_k, q_k)$ and $\delta(q_{\overline{k}}, op_k) = (r_{\overline{k}}, q)$. Thus, $e_0 = (P_0, op_0, \mathcal{O}, r_0)$ and $e_1 = (P_1, op_1, \mathcal{O}, r_1)$, and they are applicable to $e_1(\mathcal{C})$ and $e_0(\mathcal{C})$, respectively. Also, $\mathcal{O}$ has the same state, namely q, in both $e_0(e_1(\mathcal{C}))$ and $e_1(e_0(\mathcal{C}))$. Thus, the state of each process and each object is in the same state in $e_0(e_1(\mathcal{C}))$ as in $e_1(e_0(\mathcal{C}))$. By Lemma 4, $e_0(e_1(\mathcal{C}))$ and $e_1(e_0(\mathcal{C}))$ have the same valence. This contradicts the fact that $e_0(e_1(\mathcal{C}))$ and $e_1(e_0(\mathcal{C}))$ have opposite valence (because $e_1(\mathcal{C})$ and $e_0(\mathcal{C})$ do).

Neither one of op_0 and op_1 overwrites the other at q^*: Suppose, for contradiction, that op_k overwrites $op_{\overline{k}}$, for some $k \in \{0,1\}$. Then, there exist responses r_0, r_1 and states s, q such that $\delta(q^*, op_{\overline{k}}) = (r_{\overline{k}}, s)$ and $\delta(q^*, op_k) = \delta(s, op_k) = (r_k, q)$. Thus, $e_k = (P_k, op_k, \mathcal{O}, r_k)$ and $e_{\overline{k}} = (P_{\overline{k}}, op_{\overline{k}}, \mathcal{O}, r_{\overline{k}})$, and e_k is applicable to $e_{\overline{k}}(\mathcal{C})$. After e_k is applied to $e_{\overline{k}}(\mathcal{C})$, it is easy to see that the state of P_k and each object (including $\mathcal{O}$) is the same in $e_k(e_{\overline{k}}(\mathcal{C}))$ as in $e_k(\mathcal{C})$. By Lemma 4, $e_k(e_{\overline{k}}(\mathcal{C}))$ and $e_k(\mathcal{C})$ have the same valence. This contradicts the fact that $e_k(e_{\overline{k}}(\mathcal{C}))$ is $\overline{k}$-valent (because $e_{\overline{k}}(\mathcal{C})$ is) while $e_k(\mathcal{C})$ is k-valent. □

Lemma 8 *Object $\mathcal{O}$ is of type* dor*; furthermore, no operation pending in $\mathcal{C}$ is stationary at q^*.*

Proof. If $\mathcal{O}$ was a register, then each of the operations of P_0 and P_1 pending in $\mathcal{C}$ is either a read or a write to $\mathcal{O}$. For each of the four possible combinations, it is straightforward to verify that the two operations commute or one of them overwrites the other, contradicting Lemma 7. Thus, $\mathcal{O}$ is a dor object.

If, for some $k \in \{0,1\}$, the operation of P_k pending in $\mathcal{C}$ is stationary at q^*, then the operation of $P_{\overline{k}}$ pending in $\mathcal{C}$ overwrites P_k's operation at q^*. But this is not possible, by Lemma 7. □

From the specification of dor, it is easy to verify that every operation is stationary at ☹, and that the REVEAL operation is stationary at every state of dor. By Lemma 8, $q^* \neq$ ☹ and both processes have an ENROLL operation to $\mathcal{O}$ pending in $\mathcal{C}$. Therefore, e_0 and e_1 are ENROLL steps to $\mathcal{O}$, and thus are applicable to $e_1(\mathcal{C})$ and $e_0(\mathcal{C})$, respectively. For each $k \in \{0,1\}$, let v_k be the parameter of P_k's ENROLL operation pending in $\mathcal{C}$.

Lemma 9 *If q^* is normal, then*

(a) For each $k \in \{0,1\}$, ENROLL(v_k) is not upsetting at q^*.
(b) $v_0 \neq v_1$.

Proof. (a) Suppose, for contradiction, that ENROLL(v_k) is upsetting at q^* for some $k \in \{0,1\}$. Without loss of generality, assume that $k = 0$. By Lemma 8, ENROLL(v_1) cannot be stationary in q^*, and therefore there are two remaining cases:

Case 1. $\text{ENROLL}(v_1)$ *is upsetting at* q^*: Then, $\text{ENROLL}\ (v_0)$ and $\text{ENROLL}(v_1)$ overwrite each other at q^*. This contradicts Lemma 7.

Case 2. $\text{ENROLL}(v_1)$ *is ordinary at* q^*: Let q be the state of $\mathcal{O}$ in $e_1(\mathcal{C})$. Since $\text{ENROLL}(v_0)$ is upsetting at q^*, it is easy to verify from the specification of dor that $\text{ENROLL}(v_0)$ is also upsetting at q. But then $\text{ENROLL}(v_0)$ overwrites $\text{ENROLL}(v_1)$ at q^*. This again contradicts Lemma 7.

Since both cases contradict Lemma 7, no ENROLL operation to $\mathcal{O}$ pending in $\mathcal{C}$ is upsetting at q^*, as wanted.

(b) Note that, for any state q of dor and $v \in \{0, \ldots, 5\}$, two $\text{ENROLL}(v)$ operations commute at q. If, contrary to this part of the lemma, $v_0 = v_1 = v$, for some $v \in \{0, \ldots, 5\}$, the two $\text{ENROLL}(v)$ operations pending in $\mathcal{C}$ commute at q^*, contrary to Lemma 7. Thus, $v_0 \neq v_1$. □

In the rest of this section, let $\mathcal{C}_0 = e_1(e_0(\mathcal{C}))$ and $\mathcal{C}_1 = e_0(e_1(\mathcal{C}))$.

Lemma 10 $q^* = ☺$.

Proof. By Lemma 8, $q^* \neq ☹$. Thus, it suffices to show that q^* is not normal. By Lemmata 7–9, if q^* is a normal state, the following properties hold: The two ENROLL operations pending in $\mathcal{C}$ have distinct parameters, and they are ordinary, do not commute, and neither one of them overwrites the other at q^*. By inspection of Fig. 2, it can be verified that no normal state except N_0 satisfies this property. (See Fig. 5 for the details of this verification.) We now eliminate the remaining possibility, i.e., that $q^* = N_0$.

By the specification of dor, $\text{ENROLL}(v)$ is ordinary at N_0 for $v \in \{1, 2, 3, 4\}$. Thus, by Lemma 8 and Lemma 9(a), $v_0, v_1 \in \{1, 2, 3, 4\}$. First we prove that $v_0, v_1 \neq 1$. Suppose to the contrary that $v_k = 1$ for some $k \in \{0, 1\}$. Then, by Lemma 9(b), $v_{\overline{k}} \in \{2, 3, 4\}$. By inspection of the graph in Fig. 2, at state N_0, $\text{ENROLL}(2)$ and $\text{ENROLL}(1)$ commute, $\text{ENROLL}(3)$ overwrites $\text{ENROLL}(1)$, and $\text{ENROLL}(4)$ overwrites $\text{ENROLL}(1)$. Each of these three cases contradicts Lemma 7. Hence, $v_0, v_1 \neq 1$.

Next we show that $v_0, v_1 \neq 4$. Again suppose, for contradiction, that $v_k = 4$ for some $k \in \{0, 1\}$. Then $v_{\overline{k}} \in \{2, 3\}$ (by Lemma 9(b) and the claim in the preceding paragraph). By inspection of Fig. 2, $\text{ENROLL}(4)$ commutes with both $\text{ENROLL}(2)$ and $\text{ENROLL}(3)$ at N_0. This contradicts Lemma 7.

We have shown that $v_0, v_1 \notin \{1, 4\}$. Thus, $v_0, v_1 \in \{2, 3\}$ and, by Lemma 9(b), they are distinct. Without loss of generality assume that $v_0 = 2$ and $v_1 = 3$. (The other case, where $v_0 = 3$ and $v_1 = 2$, is similar.) Thus, $e_0 = (P_0, \text{ENROLL}(2), \mathcal{O}, ack)$ and $e_1 = (P_1, \text{ENROLL}(3), \mathcal{O}, ack)$. Note that $\mathcal{C}_0$ and $\mathcal{C}_1$ are identical, except the state of $\mathcal{O}$ in the former is N_{023} while in the latter it is N_{0123}.

Claim 10.1 There is a schedule $\hat{S}$ such that (a) $\hat{S}$ is applicable to both $\mathcal{C}_0$ and $\mathcal{C}_1$ (b) the state of each process and each object except $\mathcal{O}$ is the same in $\hat{S}(\mathcal{C}_0)$ as in $\hat{S}(\mathcal{C}_1)$, (c) each process has an ENROLL operation to $\mathcal{O}$ pending in both $\hat{S}(\mathcal{C}_0)$ and $\hat{S}(\mathcal{C}_1)$, and (d) the state of $\mathcal{O}$ in $\hat{S}(\mathcal{C}_0)$ is N_{023} while in $\hat{S}(\mathcal{C}_1)$ it is N_{0123}.

Proof of Claim 10.1. By Lemma 3, P_0 has a solo schedule S applicable to $\mathcal{C}_0$ such that P_0 has decided in $S(\mathcal{C}_0)$. Since $\mathcal{C}_0$ is 0-valent, P_0 decides 0 in $S(\mathcal{C}_0)$. We claim that S has an ENROLL step to $\mathcal{O}$. If not, since N_{023} and N_{0123} (the states of $\mathcal{O}$ in $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively) are both white and since REVEAL (the only other kind of operation that may be applied to $\mathcal{O}$) is stationary at these states, then by Lemma 5, S is applicable to $\mathcal{C}_1$ and P_0 has the same state in $S(\mathcal{C}_0)$ as in $S(\mathcal{C}_1)$. This implies P_0 also decides 0 in $S(\mathcal{C}_1)$, which contradicts the fact that $\mathcal{C}_1$ is 1-valent. Thus, S contains an ENROLL step to $\mathcal{O}$, as claimed.

Let S_0 be the longest prefix of S that does not contain an ENROLL step to $\mathcal{O}$. Again by Lemma 5, S_0 is applicable to $\mathcal{C}_1$, the state of each process and each object except $\mathcal{O}$ is the same in $S_0(\mathcal{C}_0)$ as in $S_0(\mathcal{C}_1)$, and the state of $\mathcal{O}$ in $S_0(\mathcal{C}_0)$ is N_{023} while in $S_0(\mathcal{C}_1)$ it is N_{0123}. Clearly, by the definition of S_0, the operation of P_0 pending in both $S_0(\mathcal{C}_0)$ and $S_0(\mathcal{C}_1)$ is an ENROLL to $\mathcal{O}$.

Applying a similar argument (replacing P_0 with P_1) to configurations $S_0(\mathcal{C}_0)$ and $S_0(\mathcal{C}_1)$, we can show that there exists a solo schedule S_1 of P_1 such that S_1 is applicable to both $S_0(\mathcal{C}_0)$ and $S_0(\mathcal{C}_1)$, the state of each process and each object except $\mathcal{O}$ is the same in $S_1(S_0(\mathcal{C}_0))$ as in $S_1(S_0(\mathcal{C}_1))$, P_1 has an ENROLL to $\mathcal{O}$ pending in both $S_1(S_0(\mathcal{C}_0))$ and $S_1(S_0(\mathcal{C}_1))$, and the state of $\mathcal{O}$ in $S_1(S_0(\mathcal{C}_0))$ is N_{023} while in $S_1(S_0(\mathcal{C}_1))$ it is N_{0123}. Recall that P_0 has an ENROLL to $\mathcal{O}$ pending in $S_0(\mathcal{C}_0)$ and $S_0(\mathcal{C}_1)$. Since S_1 is a solo schedule of P_1, the operation of P_0 pending in both $S_1(S_0(\mathcal{C}_0))$ and $S_1(S_0(\mathcal{C}_1))$ is still an ENROLL to $\mathcal{O}$. Thus, $\hat{S} = S_0 \cdot S_1$ has the properties stated in the claim. □ Claim 10.1

For each $v \in \{1, 2, 3, 4\}$, $\text{ENROLL}(v)$ is ordinary at N_V if $v \notin V$. Also, $\text{ENROLL}(5)$ is ordinary at N_V if $5 \notin V$ and 2 and at least one of $\{3, 4\}$ are in V.

1. N_{01}: Only $\text{ENROLL}(2)$, $\text{ENROLL}(3)$ and $\text{ENROLL}(4)$ are ordinary at N_{01}, and any two of these three commute.
2. N_{02}: Only $\text{ENROLL}(1)$, $\text{ENROLL}(3)$ and $\text{ENROLL}(4)$ are ordinary at N_{02}; and $\text{ENROLL}(4)$ overwrites $\text{ENROLL}(1)$, while $\text{ENROLL}(3)$ commutes with $\text{ENROLL}(1)$ and $\text{ENROLL}(4)$.
3. N_{012}: Only $\text{ENROLL}(3)$ and $\text{ENROLL}(4)$ are ordinary at N_{012}, and they commute.
4. N_{013}: Only $\text{ENROLL}(2)$ and $\text{ENROLL}(4)$ are ordinary at N_{013}, and they commute.
5. N_{014}: Only $\text{ENROLL}(2)$ and $\text{ENROLL}(3)$ are ordinary at N_{014}, and they commute.
6. N_{023}: Only $\text{ENROLL}(1)$, $\text{ENROLL}(4)$ and $\text{ENROLL}(5)$ are ordinary at N_{023}; and $\text{ENROLL}(4)$ overwrites $\text{ENROLL}(1)$, while $\text{ENROLL}(5)$ commutes with $\text{ENROLL}(1)$ and $\text{ENROLL}(4)$.
7. N_{0123}: Only $\text{ENROLL}(4)$ and $\text{ENROLL}(5)$ are ordinary at N_{0123}, and they commute.
8. N_{0124}: Only $\text{ENROLL}(3)$ and $\text{ENROLL}(5)$ are ordinary at N_{0124}, and they commute.
9. N_{0134}: Only $\text{ENROLL}(2)$ is ordinary at N_{0134}.
10. N_{0235}: Only $\text{ENROLL}(1)$ and $\text{ENROLL}(4)$ are ordinary at N_{0235}, and $\text{ENROLL}(4)$ overwrites $\text{ENROLL}(1)$.
11. N_{01234}: Only $\text{ENROLL}(5)$ is ordinary at N_{01234}.
12. N_{01235}: Only $\text{ENROLL}(4)$ is ordinary at N_{01235}.
13. N_{01245}: Only $\text{ENROLL}(3)$ is ordinary at N_{01245}.
14. N_{012345}: There is no ordinary ENROLL operation at N_{012345}.

Fig. 5. Normal states that cannot be q^*

Let $A_0 = \hat{S}(\mathcal{C}_0)$ and $A_1 = \hat{S}(\mathcal{C}_1)$, and let w_0 and w_1 be the parameters of the ENROLL operations of P_0 and P_1, respectively, pending in both A_0 and A_1. For each $k \in \{0,1\}$, let $c_k = (P_k, \text{ENROLL}(w_k), \mathcal{O}, ack)$; i.e., c_k is the step of P_k that is applicable to both A_0 and A_1. There are four cases:

Case 1. $w_k \in \{0,2,3\}$, *for some* $k \in \{0,1\}$: Then $\text{ENROLL}(w_k)$ is upsetting in both N_{023} and N_{0123} (see Fig. 2). Since $\mathcal{O}$ has states N_{023} and N_{0123} in configurations A_0 and A_1, respectively, it follows that $\mathcal{O}$ is in state ☹ in both $c_k(A_0)$ and $c_k(A_1)$. Then, the state of P_k and each object (including $\mathcal{O}$) in $c_k(A_0)$ is the same as in $c_k(A_1)$. By Lemma 4, $c_k(A_0)$ and $c_k(A_1)$ have the same valence. This contradicts the fact that $c_k(A_0)$ is 0-valent and $c_k(A_1)$ is 1-valent.

Case 2. $w_k = 4$, *for some* $k \in \{0,1\}$: In this case, the state of $\mathcal{O}$ in both $c_k(A_0)$ and $c_k(A_1)$ is N_{01234} (see Fig. 2). Thus, the state of P_k and each object is the same in $c_k(A_0)$ as in $c_k(A_1)$. By Lemma 4, $c_k(A_0)$ and $c_k(A_1)$ have the same valence. which is a contradiction as in Case 1.

Case 3. $w_k = 1$, *for some* $k \in \{0,1\}$: In this case, $\mathcal{O}$ is in the same state, namely N_{0123}, in both $c_k(A_0)$ and A_1 (see Fig. 2). The state of $P_{\overline{k}}$ and each object is the same in $c_k(A_0)$ as in A_1. By Lemma 4, $c_k(A_0)$ and A_1 have the same valence. This contradicts the fact that $c_k(A_0)$ is 0-valent while A_1 is 1-valent.

Case 4. $w_0 = w_1 = 5$: Then, $\mathcal{O}$ is in the same state, namely ☹, in both $c_1(c_0(A_0))$ and $c_1(c_0(A_1))$ (see Fig. 2). It follows that every process and every object is in the same state in $c_1(c_0(A_0))$ as in $c_1(c_0(A_1))$. This contradicts Lemma 4, since the nodes $c_1(c_0(A_0))$ and $c_1(c_0(A_1))$ have opposite valence.

Each of these cases leads to a contradiction. Thus $q^* \neq N_0$, and the lemma follows. □

By the specification of dor (see Fig. 2), every operation – except $\text{ENROLL}(0)$ and $\text{ENROLL}(1)$ – is stationary at ☺. Thus, by Lemma 8, the two operations to $\mathcal{O}$ pending in $\mathcal{C}$ must be either $\text{ENROLL}(0)$ or $\text{ENROLL}(1)$. Furthermore, one of these must be $\text{ENROLL}(0)$, while the other is $\text{ENROLL}(1)$. (If not, since two $\text{ENROLL}(0)$'s or two $\text{ENROLL}(1)$'s commute at ☺, then the two operations pending in $\mathcal{C}$ commute at ☺, which contradicts Lemma 7.) Without loss of generality, we assume that P_0 and P_1 have $\text{ENROLL}(0)$ and $\text{ENROLL}(1)$ to $\mathcal{O}$, respectively, pending in $\mathcal{C}$. (If not, we simply interchange the roles of P_0 and P_1, as well as of e_0 and e_1, throughout the argument presented below.) Thus, $e_0 = (P_0, \text{ENROLL}(0), \mathcal{O}, ack)$ and $e_1 = (P_1, \text{ENROLL}(1), \mathcal{O}, ack)$, and therefore $\mathcal{C}_0$ and $\mathcal{C}_1$ differ only in that the state of $\mathcal{O}$ in $\mathcal{C}_0$ is N_{01} while in $\mathcal{C}_1$ it is ☹.

Lemma 11 *There exists a solo schedule E_0 of process P_0 such that*

- *E_0 is applicable to both $\mathcal{C}_0$ and $\mathcal{C}_1$;*
- *$E_0(\mathcal{C}_0)$ and $E_0(\mathcal{C}_1)$ are identical, except the state of $\mathcal{O}$ in $E_0(\mathcal{C}_0)$ is N_{01}, while in $E_0(\mathcal{C}_1)$ it is ☹; and*
- *P_0 has an $\text{ENROLL}(2)$ to $\mathcal{O}$ pending in both $E_0(\mathcal{C}_0)$ and $E_0(\mathcal{C}_1)$.*

Proof. Consider the configuration $e_0(\mathcal{C})$. By Lemma 3, P_0 has a solo schedule S applicable to $e_0(\mathcal{C})$ such that P_0 has decided in $S(e_0(\mathcal{C}))$. Since $e_0(\mathcal{C})$ is 0-valent, P_0 decides 0 in $S(e_0(\mathcal{C}))$. For each $k \in \{0,1\}$, $e_k = (P_k, \text{ENROLL}(k), \mathcal{O}, ack)$.

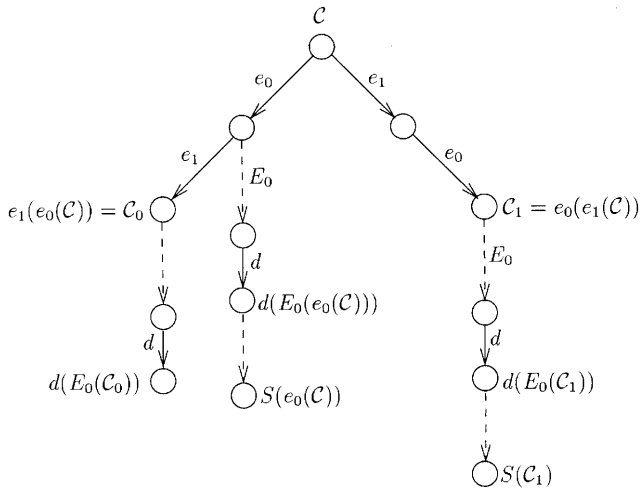


Fig. 6. A solo schedule S of P_0 that is applicable to $\mathcal{C}$

(See Fig. 6.) We claim that S contains an $\text{ENROLL}(v)$ step to $\mathcal{O}$ for some $v \in \{0, \ldots, 4\}$. To prove this claim we suppose, for contradiction, that S does not contain such an ENROLL step. First, note that process P_0 and each object except $\mathcal{O}$ is in the same state in $e_0(\mathcal{C})$ as in $\mathcal{C}_1$ (recall that $\mathcal{C}_1 = e_0(e_1(\mathcal{C}))$). Also, the state of $\mathcal{O}$ in $e_0(\mathcal{C})$ is N_0 while in $\mathcal{C}_1$ it is ☹; thus, $\mathcal{O}$ is in a white state in both $e_0(\mathcal{C})$ and $\mathcal{C}_1$. Since $\text{ENROLL}(5)$ and REVEAL (the only operations that may be applied to $\mathcal{O}$ in S) are stationary at both N_0 and ☹, it follows from Lemma 5 (with $C = e_0(\mathcal{C})$ and $C' = \mathcal{C}_1$) that S is applicable to $\mathcal{C}_1$ and P_0 has the same state in $S(e_0(\mathcal{C}))$ as in $S(\mathcal{C}_1)$. This implies that P_0 also decides 0 in $S(\mathcal{C}_1)$, which contradicts the fact that $\mathcal{C}_1$ is 1-valent. Hence, S contains an $\text{ENROLL}(v)$ step to $\mathcal{O}$, for some $v \in \{0, \ldots, 4\}$, as claimed.

Let $E_0 \cdot d$ be the shortest prefix of S that contains an ENROLL step to $\mathcal{O}$ other than $\text{ENROLL}(5)$. Thus, $d = (P_0, \text{ENROLL}(v), \mathcal{O}, ack)$ for some $v \in \{0, \ldots, 4\}$, and E_0 contains no ENROLL step to $\mathcal{O}$ except $\text{ENROLL}(5)$. Since N_0, N_{01} and ☹ (the states of $\mathcal{O}$ in $e_0(\mathcal{C})$, $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively) are white and both $\text{ENROLL}(5)$ and REVEAL (the only operations that may be applied to $\mathcal{O}$ in E_0) are stationary at these states, it follows from Lemma 5 (with $C = e_0(\mathcal{C})$, $C' \in \{\mathcal{C}_0, \mathcal{C}_1\}$, and $S = E_0$) that

- E_0 is applicable to both $\mathcal{C}_0$ and $\mathcal{C}_1$,
- the state of P_0 and each object except $\mathcal{O}$ is the same in $E_0(\mathcal{C}_0)$ as in $E_0(\mathcal{C}_1)$, and
- the state of $\mathcal{O}$ in $E_0(\mathcal{C}_0)$ is N_{01}, while in $E_0(\mathcal{C}_1)$ it is ☹.

Clearly, P_1 has the same state in $E_0(\mathcal{C}_0)$ as in $E_0(\mathcal{C}_1)$ (since E_0 is a solo schedule of P_0). To complete the proof of this lemma, it remains to show that $v = 2$. For this, it suffices to preclude the possibility that $v \in \{0,1,3,4\}$. Let $A_0 = d(E_0(e_0(\mathcal{C})))$ and $A_1 = d(E_0(\mathcal{C}_1))$.

Case 1. $v = 0$: Since $\mathcal{O}$ is in state N_0 in $E_0(e_0(\mathcal{C}))$ and $\text{ENROLL}(0)$ is upsetting at N_0 (see Fig. 2), $\mathcal{O}$ is in state ☹ in A_0. Since $\mathcal{O}$ is in state ☹ in $E_0(\mathcal{C}_1)$, it remains in that state in A_1. Thus, the state of P_0 and each object is the same in A_0 as in A_1. By Lemma 4, A_0 and A_1 have the same valence. This is a contradiction, since A_0 and A_1 have opposite valence.

Case 1. $v \in \{1,3,4\}$: Since $\mathcal{O}$ has state N_0 in $E_0(e_0(\mathcal{C}))$, the state of $\mathcal{O}$ in A_0 is N_{01} if $v = 1$, N_{013} if $v = 3$, and N_{014} if $v = 4$ (see Fig. 2). Recall that P_1 has an $\text{ENROLL}(1)$ to $\mathcal{O}$ pending in $e_0(\mathcal{C})$. Consequently, as $E_0 \cdot d$ is a solo schedule of P_0, the operation of P_1 pending in A_0 is still $\text{ENROLL}(1)$ to $\mathcal{O}$. Thus, $e_1 = (P_1, \text{ENROLL}(1), \mathcal{O}, ack)$ is the step of P_1 that is applicable to A_0. By the specification of dor, $\text{ENROLL}(1)$ is upsetting at each of $\{N_{01}, N_{013}, N_{014}\}$. It follows that $\mathcal{O}$ is in state ☹ in $e_1(A_0)$. Then, each process and object (including $\mathcal{O}$) has the same state in $e_1(A_0)$ as in A_1. By Lemma 4, $e_1(A_0)$ and and A_1 have the same valence. This contradicts the fact that $e_1(A_0)$ is 0-valent and A_1 is 1-valent. □

Lemma 12 *Let E_0 be as in Lemma 11. There exists a solo schedule E_1 of process P_1 such that*

- *E_1 is applicable to $E_0(\mathcal{C}_0)$ and $E_0(\mathcal{C}_1)$;*
- *$\mathcal{B}_0 = E_1(E_0(\mathcal{C}_0))$ and $\mathcal{B}_1 = E_1(E_0(\mathcal{C}_1))$ are identical, except the state of $\mathcal{O}$ in $\mathcal{B}_0$ is N_{01} while in $\mathcal{B}_1$ it is ☹; and*
- *for some $v \in \{3,4\}$, P_1 has an $\text{ENROLL}(v)$ to $\mathcal{O}$ pending in both $\mathcal{B}_0$ and $\mathcal{B}_1$.*

Proof. Consider the configuration $E_0(\mathcal{C}_0)$. By Lemma 3, process P_1 has a solo schedule S_1 applicable to $E_0(\mathcal{C}_0)$ such that P_1 has decided in $S(E_0(\mathcal{C}_0))$. By Lemma 5, using an argument similar to that in Lemma 11, we can show that S has a prefix $E_1 \cdot d$ such that $d = (P_1, \text{ENROLL}(v), \mathcal{O}, ack)$ for some $v \in \{0, \ldots, 4\}$, E_1 is applicable to both $E_0(\mathcal{C}_0)$ and $E_0(\mathcal{C}_1)$, and $\mathcal{B}_0 = E_1(E_0(\mathcal{C}_0))$ and $\mathcal{B}_1 = E_1(E_0(\mathcal{C}_1))$ are identical, except that the state of $\mathcal{O}$ in $\mathcal{B}_0$ is N_{01} while in $\mathcal{B}_1$ it is ☹. To complete the proof it remains to show that $v \in \{3,4\}$. For this, it suffices to preclude the possibility that $v \in \{0,1,2\}$.

Case 1. $v \in \{0,1\}$: Since $\mathcal{O}$ is in state N_{01} in $\mathcal{B}_0$ and both $\text{ENROLL}(0)$ and $\text{ENROLL}(1)$ are upsetting at N_{01} (see Fig. 2), object $\mathcal{O}$ is in state ☹ in $d(\mathcal{B}_0)$. Clearly, as $\mathcal{O}$ is in state ☹ in $\mathcal{B}_1$, it remains in state ☹ in $d(\mathcal{B}_1)$. Therefore, each process and object is in the same state in $d(\mathcal{B}_0)$ as in $d(\mathcal{B}_1)$. By Lemma 4, $d(\mathcal{B}_0)$ and $d(\mathcal{B}_1)$ have the same valence. This contradicts the fact that $d(\mathcal{B}_0)$ is 0-valent and $d(\mathcal{B}_1)$ is 1-valent.

Case 1. $v = 2$: In this case, $\mathcal{O}$ is in state N_{012} in $d(\mathcal{B}_0)$ (see Fig. 2). By Lemma 11, P_0 has $\text{ENROLL}(2)$ to $\mathcal{O}$ pending in both $E_0(\mathcal{C}_0)$ and $E_0(\mathcal{C}_1)$. Since $E_1 \cdot d$ is a solo schedule of P_1, the operation of P_0 pending in both $d(\mathcal{B}_0)$ and $d(\mathcal{B}_1)$ is still $\text{ENROLL}(2)$ to $\mathcal{O}$. Let $d' = (P_0, \text{ENROLL}(2), \mathcal{O}, ack)$ denote the step of P_0 that is applicable to both $d(\mathcal{B}_0)$ and $d(\mathcal{B}_1)$.

Since $\text{ENROLL}(2)$ is upsetting at N_{012} (the state of $\mathcal{O}$ in $d(\mathcal{B}_0)$), it follows that $\mathcal{O}$ is in state ☹ in $d'(d(\mathcal{B}_0))$. Because $\mathcal{O}$ is in state ☹ in $e_1(\mathcal{C})$, it is also in that state in $d'(d(\mathcal{B}_1))$. Thus, each process and object is in the same state in $d'(d(\mathcal{B}_0))$ as in $d'(d(\mathcal{B}_1))$. By Lemma 4, $d'(d(\mathcal{B}_0))$ and $d'(d(\mathcal{B}_1))$ have the same valence. This contradicts the fact that $d'(d(\mathcal{B}_0))$ is 0-valent and $d'(d(\mathcal{B}_1))$ is 1-valent. □

By Lemma 12, we can assume, without loss of generality, that P_1 has $\text{ENROLL}(3)$ to $\mathcal{O}$ pending in both $\mathcal{B}_0$ and $\mathcal{B}_1$. (If the operation of P_1 to $\mathcal{O}$ pending in $\mathcal{B}_0$ and $\mathcal{B}_1$ is an $\text{ENROLL}(4)$, instead of $\text{ENROLL}(3)$, we interchange the roles of $\text{ENROLL}(3)$ and $\text{ENROLL}(4)$, and replace states N_{013} and N_{0123} with states N_{014} and N_{0124}, respectively, throughout the argument presented below.)

For each $k \in \{0,1\}$, $e_k = (P_k, \text{ENROLL}(k), \mathcal{O}, ack)$.

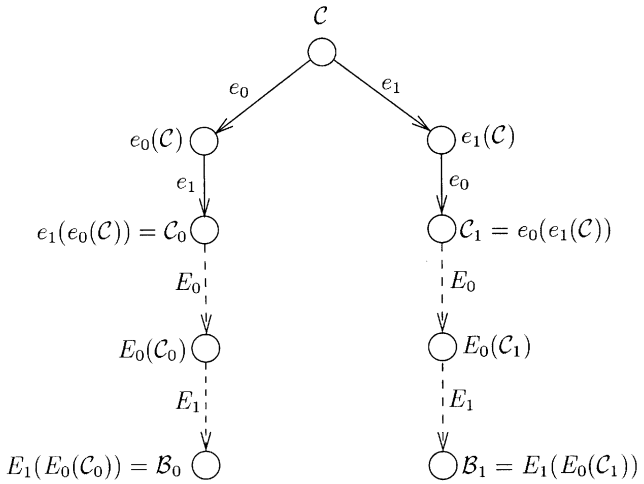


Fig. 7. Some configurations reachable from $\mathcal{C}$

To summarise, $\mathcal{B}_0$ and $\mathcal{B}_1$ satisfy the following properties: They are 0-valent and 1-valent configurations, respectively, reachable from $\mathcal{C}$ and differ only in that the state of $\mathcal{O}$ in $\mathcal{B}_0$ is N_{01} while in $\mathcal{B}_1$ it is ☹. Furthermore, for each $k \in \{0,1\}$, process P_k has an $\text{ENROLL}(k+2)$ to $\mathcal{O}$ pending in both $\mathcal{B}_0$ and $\mathcal{B}_1$. (See Fig. 7.)

We now prove some properties of schedules that are applicable to $\mathcal{B}_0$.

Lemma 13 *Let S be any schedule that is applicable to $\mathcal{B}_0$. For each $k \in \{0,1\}$, the first step taken by P_k in S is an $\text{ENROLL}(k+2)$ to object $\mathcal{O}$.*

Proof. Immediate from the fact that process P_k has an $\text{ENROLL}(k+2)$ to $\mathcal{O}$ pending in both $\mathcal{B}_0$ and $\mathcal{B}_1$, for each $k \in \{0,1\}$. □

Let S be any schedule of algorithm **A** that is applicable to $\mathcal{B}_0$. We say that S is *deciding* if it contains either

- an $\text{ENROLL}(4)$ step to $\mathcal{O}$, or
- an $\text{ENROLL}(5)$ step to $\mathcal{O}$ by a process that appears after a step in S by the other process.

S is *nondeciding* if it is not deciding.

Lemma 14 *Let S be any schedule that is applicable to $\mathcal{B}_0$. If S is nondeciding then*

(a) S is applicable to $\mathcal{B}_1$, and the state of each process and each object except $\mathcal{O}$ is the same in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$; and

(b) S contains at most one $\text{ENROLL}(2)$ step to $\mathcal{O}$, at most one $\text{ENROLL}(3)$ step to $\mathcal{O}$, and no $\text{ENROLL}(v)$ step to $\mathcal{O}$ for any $v \in \{0,1\}$.

Proof. (a) By Lemma 12, $\mathcal{B}_0$ and $\mathcal{B}_1$ differ only in that the state of $\mathcal{O}$ in the former is N_{01}, while in the latter it is ☹. As mentioned, both N_{01} and ☹ are white. Since S is nondeciding, S does not contain an $\text{ENROLL}(4)$ step to $\mathcal{O}$, or an $\text{ENROLL}(5)$ step to $\mathcal{O}$ by a process that appears after a step in S by the other process. Recall that, for each $k \in \{0,1\}$, P_k has an $\text{ENROLL}(k+2)$ to $\mathcal{O}$ pending in $\mathcal{B}_0$. Therefore, S does not

contain an $\text{ENROLL}(5)$ step to $\mathcal{O}$ that appears after both an $\text{ENROLL}(2)$ and an $\text{ENROLL}(3)$ step to $\mathcal{O}$. Given these facts, it is easy to verify (using Fig. 2) that for every prefix S' of S, the state of $\mathcal{O}$ in $S'(\mathcal{B}_0)$ is one of $\{N_{01}, N_{012}, N_{013}, N_{0123}, ☹\}$. Since all these states are white and every operation of type dor is stationary at ☹ (the state of $\mathcal{O}$ in $\mathcal{B}_1$), by Lemma 5, S is applicable to $\mathcal{B}_1$, and the state of each process and each object except $\mathcal{O}$ is the same in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$.

(b) Suppose, for contradiction, that S contains more than one $\text{ENROLL}(2)$ step to $\mathcal{O}$, or more than one $\text{ENROLL}(3)$ step to $\mathcal{O}$, or an $\text{ENROLL}(v)$ step to $\mathcal{O}$ for some $v \in \{0, 1\}$. By inspection of Fig. 2, it is easy to verify that, under this assumption, the state of $\mathcal{O}$ in $S(\mathcal{B}_0)$ is ☹. By (a), S is applicable to $\mathcal{B}_1$ and the state of each process and each object except $\mathcal{O}$ is the same in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$. Since $\mathcal{O}$ is in state ☹ in $\mathcal{B}_1$, it remains in state ☹ in $S(\mathcal{B}_1)$. Thus, the state of each process and object (including $\mathcal{O}$) is the same in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$. By Lemma 4, $S(\mathcal{B}_0)$ and $S(\mathcal{B}_1)$ have the same valence. This contradicts the fact that $S(\mathcal{B}_0)$ is 0-valent and $S(\mathcal{B}_1)$ is 1-valent. □

The next two lemmata show that the configuration $\mathcal{B}_0$ and the dor object $\mathcal{O}$ satisfy the properties stated at the beginning of this section. That is, starting from $\mathcal{B}_0$, no process can decide unless at least one of them has applied to $\mathcal{O}$ either an $\text{ENROLL}(4)$, or an $\text{ENROLL}(5)$ *after* the other process has taken at least one step (Lemma 15). Furthermore, if *both* processes are about to execute the $\text{ENROLL}(4)$ or $\text{ENROLL}(5)$ operation to $\mathcal{O}$ that they must execute before they can decide, then one of the two must execute an $\text{ENROLL}(4)$, and the other must execute an $\text{ENROLL}(5)$: they can't both be about to execute the same operation (Lemma 16).

Lemma 15 *Let S be any finite schedule that is applicable to $\mathcal{B}_0$ such that some process has decided in $S(\mathcal{B}_0)$. Then S is deciding.*

Proof. Let P be the process that has decided in $S(\mathcal{B}_0)$. Since $S(\mathcal{B}_0)$ is 0-valent, P decides 0 in $S(\mathcal{B}_0)$. If S is nondeciding, then by Lemma 14(a), S is applicable to $\mathcal{B}_1$, and the state of each process and each object except $\mathcal{O}$ is the same in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$. In particular, P has the same state in $S(\mathcal{B}_0)$ as in $S(\mathcal{B}_1)$. But then P also decides 0 in $S(\mathcal{B}_1)$. This contradicts the fact that $S(\mathcal{B}_1)$ is 1-valent. □

Lemma 16 *Let S be any schedule applicable to $\mathcal{B}_0$ such that*

- *S is nondeciding;*
- *both processes have taken steps in S; and*
- *for each $k \in \{0, 1\}$, process P_k has an $\text{ENROLL}(v_k)$ to $\mathcal{O}$ pending in $S(\mathcal{B}_0)$ for some v_k.*

Then, $v_0, v_1 \in \{4, 5\}$ and $v_0 \neq v_1$.

Proof. Since S is nondeciding and both processes have taken steps in S, by Lemma 13 and Lemma 14(b), S contains exactly one $\text{ENROLL}(2)$ step to $\mathcal{O}$, exactly one $\text{ENROLL}(3)$ step to $\mathcal{O}$, and no $\text{ENROLL}(v)$ step to $\mathcal{O}$ for any $v \in \{0, 1\}$. Then, by inspection of Fig. 2, it is easy to see that $\mathcal{O}$ must be in state N_{0123} in $S(\mathcal{B}_0)$.

For each $k \in \{0, 1\}$, let $c_k = (P_k, \text{ENROLL}(v_k), \mathcal{O}, ack)$. From the specification of dor, each of $\{\, \text{ENROLL}(v) : 0 \leq v \leq 3 \,\}$ is upsetting at N_{0123}. If $v_k \in \{0, 1, 2, 3\}$ for some $k \in \{0, 1\}$, then $\mathcal{O}$ is in state ☹ in $c_k(S(\mathcal{B}_0))$. Since $\mathcal{O}$ is also in state ☹ in $c_k(S(\mathcal{B}_1))$, it follows that each process and each object is in the same state in $c_k(S(\mathcal{B}_0))$ as in $c_k(S(\mathcal{B}_1))$. By Lemma 4, $c_k(S(\mathcal{B}_0))$ and $c_k(S(\mathcal{B}_1))$ have the same valence. This contradicts the fact that $c_k(S(\mathcal{B}_0))$ is 0-valent and $c_k(S(\mathcal{B}_1))$ is 1-valent. Therefore, $v_0, v_1 \in \{4, 5\}$.

To complete the proof it remains to show that $v_0 \neq v_1$. Since c_1 is an ENROLL step to $\mathcal{O}$, it is applicable to both $c_0(S(\mathcal{B}_0))$ and $c_0(S(\mathcal{B}_1))$. Let $A_0 = c_1(c_0(S(\mathcal{B}_0)))$ and $A_1 = c_1(c_0(S(\mathcal{B}_1)))$. If $v_0 = v_1$, then $\mathcal{O}$ is in state ☹ in both A_0 and A_1. Then, each process and each object has the same state in A_1 as in A_0. By Lemma 4, A_0 and A_1 have the same valence. This contradicts the fact that A_0 is 0-valent and A_1 is 1-valent. Hence, it must be that $v_0 \neq v_1$, as desired. □

3.3.3 Stage two of the proof

We now describe an algorithm, derived from **A**, that solves one-resilient Consensus for two processes, Q_0 and Q_1, and uses one dor object less than **A**. The idea is very similar to the algorithm used in [10]: Each process Q_k simulates the actions of the corresponding process P_k in an execution of **A**, and uses this execution to determine what value to decide. In this simulation, the processes are allowed to access all objects used by **A**, except $\mathcal{O}$; each of these objects is initialised to the state it has in configuration $\mathcal{B}_0$.

Each process Q_k starts its simulation of P_k by pretending that P_k is in the state it has in configuration $\mathcal{B}_0$. Based on the current state of (the simulated) P_k, Q_k determines the operation op that P_k would apply next, and the object O to which op would be applied. If $O \neq \mathcal{O}$, then Q_k has access to O and can apply op to it directly to determine the response and update the (simulated) state of P_k accordingly. If, however, $O = \mathcal{O}$, this is not possible, since Q_k is not allowed to access $\mathcal{O}$. Instead, in this case, Q_k acts as follows:

If op is a REVEAL *operation*, then Q_k pretends that it applied this operation to $\mathcal{O}$ and received response 1.

If op is an $\text{ENROLL}(v)$ *operation*, then there are three cases:

Case 1. $v = k + 2$: Q_k pretends that it applied this operation to $\mathcal{O}$ and received response ack.

Case 2. $v = 4$: Q_k stops simulating steps of P_k in **A** and decides its own initial value.

Case 3. $v = 5$: Q_k first checks whether $Q_{\overline{k}}$ has simulated its first step of $P_{\overline{k}}$. If not, Q_k pretends that it applied the $\text{ENROLL}(5)$ operation to $\mathcal{O}$ and received response ack. Otherwise, Q_k stops simulating steps of **A** and decides $Q_{\overline{k}}$'s initial value.

If Q_k does not decide, then it updates the (simulated) state of P_k as if op were applied to $\mathcal{O}$ (as described above) and then proceeds with the simulation of the next operation of P_k.

The algorithm is described more formally in Fig. 8. A few remarks about the conventions used in the pseudocode are in order. Recall, from Fig. 3, that $\text{Apply}(P_k, op, O)$ is the procedure by which P_k invokes operation op on object O; it returns the result of this invocation (and updates the state of O accordingly). In addition, we assume that we are given two functions

Shared: all objects used in $\mathbf{A}$ except $\mathcal{O}$,
each initialised to the state it has in $\mathcal{B}_0$
D_0, D_1: register, each initialised to $\perp$
R_0, R_1: register, each initialised to 0
S : auxiliary variable : schedule of $\mathbf{A}$ applicable to $\mathcal{B}_0$, initially empty

Code for process Q_k, $k \in \{0,1\}$

```
1   D_k := initial value of Q_k
2   state := state of P_k in B_0
3   while Q_k has not decided do
4       ⟨op, O⟩ := NextOp(P_k, state)
5       if O ≠ 𝒪 then
6           ⟦ r := Apply(P_k, op, O)
7             S := S · (P_k, op, O, r) ⟧
8             state := NextState(P_k, state, r)
9       else (* O = 𝒪 *)
10          if op = REVEAL then
11              S := S · (P_k, REVEAL, 𝒪, 1)
12              state := NextState(P_k, state, 1)
13          else if op = ENROLL(k + 2) then
14              ⟦ R_k := 1
15                S := S · (P_k, ENROLL(k + 2), 𝒪, ack) ⟧
16                state := NextState(P_k, state, ack)
17          else if op = ENROLL(5) then
18              ⟦ if R_k̄ = 0 then
19                    S := S · (P_k, ENROLL(5), 𝒪, ack) ⟧
20                    state := NextState(P_k, state, ack)
21                else
22                    decide D_k̄
23          else (* op = ENROLL(4) *)
24              decide D_k
```

Fig. 8. Solving wait-free Consensus for two processes using $\mathbf{A}$ with one fewer dor object

that describe the behaviour of processes P_0 and P_1 in $\mathbf{A}$ (these correspond to the functions ν and τ discussed when we formally defined processes in Sect. 2.2).

- NextOp(P_k, s): returns the pair $\langle op, O\rangle$, where the next operation that P_k executes in $\mathbf{A}$ when in state s is to apply op to object O.
- NextState(P_k, s, r): returns the state that P_k enters in $\mathbf{A}$ if it receives response r from the operation it invokes when in state s.

In the algorithm, register R_k, $k \in \{0,1\}$, with initial value 0 is used by process Q_k to indicate whether it has simulated a step of P_k, Specifically, Q_k writes 1 into R_k to signify that it has simulated the step of P_k that is pending in $\mathcal{B}_0$ (lines 14–15). Another shared register D_k is used by Q_k into which it writes its initial value for Consensus before it starts simulating the steps of P_k (line 1). In addition to these shared variables, we also use an auxiliary variable S whose value, informally speaking, is the schedule of steps of $\mathbf{A}$ that have been simulated by Q_0 and Q_1 so far. This variable is not needed by the algorithm, but it is useful in proving its correctness (Lemma 17). In one atomic step, besides accessing an ordinary variable, each process can also modify the auxiliary variable S. To emphasise this, we bracket with "$[\![\cdots]\!]$" the actions that correspond to an atomic step affecting both an ordinary variable and the auxiliary variable S.

Lemma 17 *If* $\mathbf{A}$ *is a wait-free Consensus algorithm for processes* P_0 *and* P_1*, then the algorithm in Fig. 8 is also a wait-free Consensus algorithm for processes* Q_0 *and* Q_1*, using one fewer* dor *object.*

Proof. It is obvious that the algorithm in Fig. 8 uses one fewer dor object than $\mathbf{A}$. Consider an arbitrary execution of the algorithm in Fig. 8. We must prove that in this execution the three properties of Consensus – Termination, Validity and Agreement – are satisfied. First we establish some facts.

Let S_i be the value of the auxiliary variable S when S contains a schedule with exactly i steps. A straightforward induction on all $i \geq 0$ such that S_i is defined shows the following invariants:

(a) S_i is applicable to $\mathcal{B}_0$.
(b) S_i is nondeciding.
(c) Let Q_k be the process that assigns S_i to S, and $state_k$ be the value to which Q_k sets its local variable $state$ just after assigning S_i to S. For all $j \geq i$, if S_j is defined and does not contain a step of P_k after its prefix S_i, then $state_k$ is the state of process P_k in configuration $S_j(\mathcal{B}_0)$ of $\mathbf{A}$.

We now turn to the proof that the three properties of a Consensus algorithm are satisfied.

Termination: Suppose, for contradiction, that Q_k is correct and never decides in the execution. This means that the **while** loop of process Q_k does not terminate. Thus, there is an infinite schedule S^* of $\mathbf{A}$ that contains infinitely many steps of P_k such that S^* is applicable to $\mathcal{B}_0$ and is nondeciding (by Invariants (a) and (b) above). By Lemma 15, P_k does not decide in $S'(\mathcal{B}_0)$, for all prefixes S' of S^*. This contradicts the fact that $\mathbf{A}$ is a wait-free Consensus algorithm for two processes (in particular, it contradicts the Termination property of $\mathbf{A}$).

Validity: In any execution of the algorithm in Fig. 8, only process Q_k writes into D_k, for each $k \in \{0,1\}$, and the value Q_k writes there is its initial value (line 1). Thus, to show the execution satisfies Validity, it suffices to show that if Q_k decides the value in D_ℓ, then Q_ℓ has previously written into D_ℓ. There are two cases:

> *Case 1. Q_k decides in line 24:* Then Q_k decides the value in D_k, into which it has previously written.
> *Case 2. Q_k decides in line 22:* Then Q_k reads $R_{\overline{k}} = 1$ in line 18. This means that $Q_{\overline{k}}$ has already executed line 14, and hence line 1 as well. Thus, $Q_{\overline{k}}$ has written into $D_{\overline{k}}$ before Q_k decides the value in $D_{\overline{k}}$.

Agreement: Suppose both Q_0 and Q_1 decide in the execution. (Otherwise, Agreement is trivially satisfied.) Let S^* be the value of S after both processes have decided. (Note that S^* is finite.) For each $k \in \{0,1\}$, let $state_k$ be the value of the local variable $state$ of Q_k when Q_k decides. By Invariants (a) and (b), S^* is applicable to $\mathcal{B}_0$ and is nondeciding. By Invariant (c), $state_k$ is the state of P_k in $S^*(\mathcal{B}_0)$, and by the algorithm in Fig. 8, Q_k decides only when the operation of P_k pending in $S^*(\mathcal{B}_0)$ is $\textsc{enroll}(v_k)$ to $\mathcal{O}$, for some v_k. Also both P_0 and P_1 have taken steps in S^* (because, by the algorithm in Fig. 8, each Q_k must simulate at least one step of P_k before deciding). We have thus shown that the hypothesis of Lemma 16 applies for $S = S^*$. Therefore, by Lemma 16, $v_0, v_1 \in \{4,5\}$ and $v_0 \neq v_1$. Thus, one of Q_0 and Q_1 decides

by line 22, while the other decides by line 24. Therefore, both processes decide the value in the same register D_ℓ, for some $\ell \in \{0, 1\}$. By Validity, this happens after Q_ℓ has written into D_ℓ. Since at most one value is written into D_ℓ, both processes decide that one value. □

Equipped with Lemma 17, we can now prove the main result of this section.

Theorem 2 *There is no wait-free Consensus algorithm for two processes using only* dor *objects and registers.*

Proof. Suppose, for contradiction, that there is a wait-free Consensus algorithm for two processes using only dor objects and registers. Using König's Lemma, it is easy to see that any wait-free Consensus algorithm that uses objects that belong to types that exhibit finite nondeterminism uses finitely many objects. Since dor and register are deterministic types (and, *a fortiori*, they exhibit finite nondeterminism), the assumed algorithm uses a finite number of dor objects. Let **A** be such an algorithm that uses a minimal number of dor objects. Since wait-free Consensus for two processes in unsolvable using registers alone [6, 12], **A** uses at least one dor object. By Lemma 17, we can construct a wait-free Consensus algorithm for two processes that uses one fewer dor object, contrary to the definition of **A**. □

4 The case of four or more processes

In this section we show that for $n \geq 4$, any set of types $\mathcal{S}$ (that includes register) strong enough to solve one-resilient Consensus among n processes is also strong enough to solve one-resilient Consensus among $n-1$ processes. This holds even if $\mathcal{S}$ may contain nondeterministic types. We prove this by showing how a one-resilient Consensus algorithm for $n \geq 4$ processes can be simulated by just three processes using only registers in addition to the objects already used by the Consensus algorithm. This simulation implies that, if $\mathcal{S}$ implements one-resilient Consensus for n processes, it also implements one-resilient Consensus for three processes; and therefore, as mentioned in Sect. 1, for any number of processes that is greater than (or equal to) 3. In particular, then, $\mathcal{S}$ implements one-resilient Consensus for $n-1$ processes.

Similar techniques were previously used by Borowsky and Gafni [1] (see also [13, 2]) and by Chandra et al. [4]. (See Sect. 5 for further comments on how our approach differs from these earlier ones.) In Sect. 4.1 we give a special implementation of type test-and-set-register that is used as a subroutine in our simulation. The simulation itself is described in Sect. 4.2.

4.1 Type test-and-set-register

The object type test-and-set-register has two states 0 and 1, and supports a single operation TEST&SET. A TEST&SET sets the state to 1 and returns the old state. Thus, the first time TEST&SET is applied to state 0, it changes the state and returns 0; thereafter, any TEST&SET returns 1 leaving the state unchanged. In the sequel we shall refer to an object of type test-and-set-register as a "TAS register".

Shared: $Active$: **array** $[0..s-1]$ of register, each is a Boolean variable, initially false
$Closed$: register, a Boolean variable, initially false
$Done$: register, a Boolean variable, initially false

TEST&SET, return 0 or 1. Code for process Q_k, $k \in \{0, \ldots, s-1\}$

```
1  if Closed then return 1
2  Closed := true
3  Active[k] := true
4  for i := k to s − 1 do
5      repeat
6          for j := 0 to k − 1 do
7              if Active[j] then
8                  Active[k] := false;  return 1
9      until i = k or not Active[i]
   (∗ Beginning of CRITICAL SECTION ∗)
10 if not Done then
11     Done  := true
12     result := 0
13 else
14     result := 1
   (∗ End of CRITICAL SECTION ∗)
15 Active[k] := false; return result
```

Fig. 9. A non-wait-free implementation of a TAS register using only registers.

Our simulation (of a Consensus algorithm for $n \geq 4$ processes by only three processes) uses a non-wait-free implementation of test-and-set-register, initialised to state 0, for three processes. The implementation is not wait-free because a correct process may initiate a TEST&SET operation but be unable to finish it because another process crashes while applying its own TEST&SET operation. The implementation, however, guarantees that process crashes can prevent *at most one* correct process from finishing its TEST&SET invocation. This property, as we shall see, is crucial to our simulation.

An implementation of TAS registers shared by $s \geq 2$ processes with this property is shown in Fig. 9. It is based on a mutual exclusion algorithm discovered independently by Burns and Lynch [3], and Lamport [9]. A shared variable $Closed$, initially false, is used to record whether a TEST&SET has already been invoked on the TAS register. To apply a TEST&SET, a process first checks if $Closed$ is true (meaning, that a TEST&SET has been invoked). If so, the process immediately returns 1. Otherwise, the process sets $Closed$ to true and tries to enter the critical section (CS) – perhaps competing with other processes that also found $Closed =$ false. The TEST&SET operation of the first process to enter the CS returns 0; all others return 1. To determine which process is the first to enter the CS, we use another shared Boolean variable $Done$ with initial value false. The first process that enters the CS finds $Done =$ false, and sets $Done$ to true before it leaves the CS.

Lemma 18 *In any execution of the algorithm in Fig. 9, it is impossible that two processes execute the critical section (lines 10–14) at the same time.*

Proof. Suppose, for contradiction, that two processes, say Q_k and Q_ℓ, are in the critical section (CS) at the same time.

Without loss of generality, assume that $k < \ell$. Since Q_ℓ enters the CS, Q_ℓ must find $Active[k] =$ false when it executes line 7 with $j = k$. So, Q_ℓ executes its line 3 before Q_k. (This is because, by our assumption that Q_k and Q_ℓ execute the CS concurrently, after setting $Active[k]$ to true in line 3, Q_k can set $Active[k]$ back to false only in line 15.) Since Q_k also enters the CS, Q_k must find $Active[\ell] =$ false when it executes line 9 with $i = \ell$. So, Q_k executes its line 3 before Q_ℓ – a contradiction. □

Consider an infinite execution E of processes $Q_0, \ldots, Q_{s-1}$ in which each of them invokes (sequentially) zero or more TEST&SET operations by executing the algorithm in Fig. 9. We say that Q is *potentially interrupting* in E if it crashes in E during its first TEST&SET invocation, after it has taken at least one step. We say that Q is *blocked* in E if it is correct in E but never returns a response to a TEST&SET invocation it has initiated. Note that a process may be blocked in E only during its *first* TEST&SET invocation (because in all subsequent TEST&SET invocations, the process finds $Closed =$ true and returns immediately in line 1). The next lemma shows that the algorithm in Fig. 9 implements a TAS register with the "limited blocking" property described at the beginning of this subsection.

Lemma 19 *Consider any infinite concurrent execution of processes $Q_0, \ldots, Q_{s-1}$ in which each of them invokes (sequentially) zero or more* TEST&SET *operations by executing the algorithm in Fig. 9. In this execution,*

(a) at most one process is blocked;
(b) if a process is blocked, then some process is potentially interrupting;
(c) if there is no potentially interrupting process, then some process' first TEST&SET *invocation returns 0; and*
(d) at most one TEST&SET *invocation returns 0.*

Proof. (a) Suppose, for contradiction, that there are two blocked processes, say Q_k and $Q_{k'}$, for some $k \neq k'$. Without loss of generality, assume that $k' < k$. By Fig. 9 and the definition of blocked process, it follows that Q_k and $Q_{k'}$ are correct and execute nonterminating **repeat** loops in lines 5–9. Furthermore, eventually $Active[k] = Active[k'] =$ true, forever. Since $k' < k$, $k' \in \{0, 1, \ldots, k-1\}$. Therefore, eventually Q_k will execute line 7 with $j = k'$ after $Active[k']$ has been permanently set to true. At that time, Q_k will execute line 8 and return, contradicting that it executes a nonterminating loop in lines 5–9.

(b) Suppose Q_k is blocked. Therefore, Q_k is correct and executes a nonterminating **repeat** loop in lines 5–9 with $i = k'$, for some $k' > k$. This means that Q_k finds $Active[k'] =$ true (in line 9) infinitely often. Hence, $Q_{k'}$ sets $Active[k'] =$ true in line 3 of its first TEST&SET invocation (in subsequent invocations it returns in line 1), and never sets it to false (if $Q_{k'}$ set $Active[k']$ to false, it would never reset it to true, so Q_k wouldn't find $Active[k'] =$ true infinitely often). By part (a), $Q_{k'}$ cannot also be blocked. Therefore, $Q_{k'}$ crashes during its first TEST&SET invocation after taking at least one step – i.e., $Q_{k'}$ is potentially interrupting.

(c) Suppose there is no potentially interrupting process. By part (b) there is no blocked process, so every process finishes its first TEST&SET invocation, if it initiates one. Let Q_k be the process with the smallest index that finds $Closed =$ false in line 1 during its first TEST&SET invocation. This invocation finishes by Q_k returning either in line 8 or in line 15. In this invocation, Q_k does not return in line 8 because it does not execute this line (by the minimality of k, Q_k must find $Active[j] =$ false in line 7, for every $j \in \{0, \ldots, k-1\}$). Thus, Q_k finishes its first TEST&SET by returning in line 15. Hence, some process (namely Q_k) executes the CS during its first TEST&SET invocation. By Lemma 18, there is a well-defined first process to enter the CS. It is easy to see that the first TEST&SET invocation of that process returns 0.

(d) Suppose, for contradiction, that two invocations of TEST&SET, say by processes Q_k and Q_ℓ, both return 0. Thus, in these invocations Q_k and Q_ℓ both execute the CS and find that $Done =$ false in line 10. By Lemma 18, one of these two processes, say Q_k, finishes the CS before the other enters the CS. Since Q_k finds $Done =$ false in line 10, it sets $Done$ to true in line 11. Since $Done$ is never set to false again, when Q_ℓ later enters the CS it will find $Done =$ true, and will return 1, not 0 – contrary to the assumption. □

4.2 The simulation algorithm

Suppose we are given an arbitrary one-resilient Consensus algorithm **A** for n processes, $P_0, P_1, \ldots, P_{n-1}$, for some $n \geq 4$. We make no assumptions about the set of base objects used by this algorithm. We show how, using **A** and only some additional registers, three processes Q_0, Q_1 and Q_2, can solve one-resilient Consensus. Let $\mathcal{Q} = \{Q_0, Q_1, Q_2\}$ and $\mathcal{P} = \{P_0, P_1, \ldots, P_{n-1}\}$.

The main idea is that the processes in $\mathcal{Q}$ will simulate the steps of the processes in $\mathcal{P}$ so that they end up simulating an execution of **A**. As soon as some $P \in \mathcal{P}$ decides in the simulated execution, the processes in $\mathcal{Q}$ will adopt P's decision and solve Consensus. Special care must be exercised to ensure that if at most one process in $\mathcal{Q}$ crashes, the simulated execution of **A** will be one in which at most one process in $\mathcal{P}$ crashes. Since **A** is one-resilient, eventually some process will decide in the simulated execution, and thus so will the correct processes in $\mathcal{Q}$.

To ensure that each step of **A** is simulated, the processes in $\mathcal{Q}$ simulate the steps of processes in $\mathcal{P}$ in round-robin fashion. Specifically, each Q_k tries to simulate the first step of P_0, the first step of P_1 and so on, until the first steps of all processes in $\mathcal{P}$ have been simulated. Then Q_k tries to simulate the second step of P_0, the second step of P_1 and so on, until the second steps of all processes in $\mathcal{P}$ have been simulated.

For the simulation to be proper, however, the processes in $\mathcal{Q}$ must coordinate to ensure that they don't repeat the simulation of a step that has already been simulated. (Such a repetition would be disastrous because then the simulated execution might not be a legitimate execution of **A**. Consider, for example, what would happen if an "increment by one" step is simulated several times!) To prevent this, we use TAS registers shared by Q_0, Q_1 and Q_2. For each i, $0 \leq i \leq n-1$, and each $r \geq 1$, there is a TAS register $TAS[i, r]$ initialised to 0, implemented using the algorithm in Fig. 9 (with $s = 3$). Before simulating the rth step of process P_i, process Q_k applies

a TEST&SET operation to $TAS[i,r]$. If Q_k wins $TAS[i,r]$ (i.e., Q_k's TEST&SET returns 0), it is the unique process responsible for simulating the rth step of P_i. Otherwise, that step has already been or will be simulated by another process in $\mathcal{Q}$, and Q_k goes on to simulate a step of the next process.

By using $TAS[i,r]$ registers implemented as discussed in Sect. 4.1, we ensure that the crash of one of the processes in $\mathcal{Q}$, in the middle of applying the TEST&SET operation to $TAS[i,r]$, will block at most one other process' execution of TEST&SET on $TAS[i,r]$ (Lemma 19(a)). Since there are three processes in $\mathcal{Q}$, there will still remain a process in $\mathcal{Q}$ which neither crashes nor is blocked by the crash. This process will continue simulating steps of the processes in $\mathcal{P}$ (except for P_i – the process in the simulation of whose step the crash occurred) until one of them decides. (This, incidentally, is the reason why we need at least three simulator processes, and the reason why this simulation cannot be made to work in the case of $n = 3$ discussed in Sect. 3.)

The simulation algorithm is shown in Fig. 10. In addition to the $TAS[i,r]$ registers already mentioned, the simulation uses the following shared objects:

- $Instr$: an array $[0..n-1]$ of n registers. $Instr[i]$ contains the index of the first instruction of P_i that has not been simulated yet.
- $State$: an array $[0..n-1]$ of n registers. $State[i]$ contains the current (simulated) state of P_i.
- $Decision$: a register. If a process Q in $\mathcal{Q}$ simulates a step of some process P in $\mathcal{P}$ as a result of which P decides, then Q writes P's decision into $Decision$.

Besides these shared objects, the simulation also uses two auxiliary variables, S and W. S is the schedule of **A** consisting of the steps simulated so far. W is a two-dimensional array that keeps track of which process in $\mathcal{Q}$ has simulated which step of each process in $\mathcal{P}$. Specifically, if the rth step of P_i has been simulated by Q_k, then $W[i,r] = k$; if the rth step of P_i has not been simulated yet, then $W[i,r] = \perp$.

The simulation of **A** by Q_0, Q_1 and Q_2 proceeds as follows. Each Q_k enters a **while** loop (lines 2–23), from which it will exit when a process in $\mathcal{P}$ has decided in the simulated execution of **A**. In successive iterations of the loop, Q_k considers the processes in $\mathcal{P}$ in round-robin fashion: P_0, P_1, ..., P_{n-1}, P_0, P_1, ... P_{n-1},.... When considering P_i, Q_k first checks to see whether it should try to simulate the "next" step of P_i, namely step $r = Instr[i]$ (line 3). It will do so if at most one process in $\mathcal{P}$ has had fewer than $r-1$ of its steps simulated (line 4). In other words, Q_k tries to keep the simulation of all but one process in $\mathcal{P}$ within one step of each other. (It cannot hope to keep the simulation of *all* processes within one step of each other, because the simulation of one of them may be interrupted due to a crash.)[4] Assuming, then, that P_i's simulation is not too far ahead of the others, Q_k attempts to simulate the rth step of P_i by first applying a TEST&SET to $TAS[i,r]$ (line 7). (For now, ignore the **coexecute** statement in lines 6–9, and replace it by line 7. We shall explain the meaning of and need for the **coexecute** statement shortly.) If

Shared: all objects used in **A**, each initialised as specified by **A**
TAS: **array** $[0..n-1, 1..\infty]$ of test-and-set-register, each implemented by the algorithm in Fig. 9
$Instr$: **array** $[0..n-1]$ of register, each initialised to 1
$State$: **array** $[0..n-1]$ of register, each initialised to $\perp$
$Decision$: register, initialised to $\perp$
S: auxiliary variable, contains a schedule of **A**, initially empty
W: auxiliary variable, contains an array $[0..n-1, 1..\infty]$ of $\{\perp, 0, 1, 2\}$, each initially $\perp$

Code for process Q_k, $k \in \{0,1,2\}$

```
1  id := 0
2  while Decision = ⊥ do
3      rd := Instr[id]
4      if |{ t : 0 ≤ t < n ∧ Instr[t] < rd }| ≤ 1 then
5          winner := 1
6          coexecute
7              winner := Apply(Q_k, TEST&SET, TAS[id, rd])
               ▮
8              repeat until Decision ≠ ⊥
9          coend
10         if winner = 0 then
               (* Q_k wins TAS[id, rd] and must simulate the
               rd th step of P_id *)
11             if rd = 1 then   (* first step of P_id *)
12                 let u be the initial value of process Q_k
13                 State[id] := initial state of P_id where P_id
                                has initial value u
14             ⟨op, O⟩ := NextOp(P_id, State[id])
15             ⟦ res := Apply(P_id, op, O)
16               W[id, rd] := k
17               S := S · (P_id, op, O, res) ⟧
18               State[id] := NextState(P_id, State[id], res)
19               if P_id has decided in State[id] then
20                   Decision := the value decided by P_id
21               else
22                   Instr[id] := rd + 1
23     id := (id + 1) mod n
24 decide Decision
```

Fig. 10. Solving one-resilient Consensus for three processes using **A**

Q_k wins $TAS[i,r]$ (line 10), it does, in fact, simulate the rth step of P_i (line 15). If this is the first step of P_i, i.e., $r = 1$, Q_k uses its own initial value as the initial value of P_i (lines 11–13). Next, Q_k updates the state of the simulated process in $State[i]$ (line 18). If the simulated step causes P_i to decide, then that decision is written into $Decision$ (line 20). Otherwise, $Instr[i]$ is incremented by one to reflect the fact that another step of P_i has been simulated (line 22), and Q_k turns its attention to the next process in round-robin order (line 23).

In addition to the notation and conventions introduced in conjunction with the simulation described in Fig. 8, we also use the construct "**coexecute** M_1 ▮ M_2 **coend**", where M_1 and M_2 are arbitrary statements. This interleaves the execution of steps of the two (potentially nonterminating) statements, until one of the two terminates. When (and if) that occurs, the

[4] Preventing processes in $\mathcal{P}$ from getting arbitrarily ahead of others during the simulation is important for keeping bounded the number of registers used in the simulation (see Lemma 20); if we do not care about this issue, we can dispense with line 4.

execution of steps of the other statement is abandoned, and the **coexecute** statement itself terminates.

We now explain the reason for using the **coexecute** statement (lines 6–9). Consider an execution of the simulation in Fig. 10 in which a process in $\mathcal{Q}$, say Q_k, crashes while it is executing its first TEST&SET on $TAS[i,r]$. Suppose that the other two processes are correct in this execution. By Lemma 19(a), the crash of Q_k may cause at most one correct process, say Q_ℓ, to be blocked in an TEST&SET invocation on $TAS[i,r]$. Therefore, the remaining correct process in $\mathcal{Q}$, say Q_m, will not be blocked in any of its TEST&SET invocations on $TAS[i,r]$. Thus, Q_m will be able to simulate steps of processes in $\mathcal{P}$ (other than P_i) until one of them decides. Q_m will then write its decision into $Decision$ and after breaking out of the **while** loop (lines 2–23), it will decide. Remember, however, that there is also a correct process in $\mathcal{Q}$, namely Q_ℓ, that is blocked in a TEST&SET invocation on $TAS[i,r]$. By applying TEST&SET to $TAS[i,r]$ as one branch of a **coexecute** statement (line 7) and checking whether $Decision \neq \perp$ in another branch (line 8), we ensure that Q_ℓ will eventually break out of the potentially nonterminating TEST&SET invocation, and will decide.

Lemma 20 *Let $\mathcal{S}$ be any set of types and* **A** *be a one-resilient Consensus algorithm for $n \geq 2$ processes $P_0, \ldots, P_{n-1}$ using $\mathcal{S}$. The algorithm in Fig. 10 is a one-resilient Consensus algorithm for three processes Q_0, Q_1, Q_2 using $\mathcal{S}$. Furthermore, if every type in $\mathcal{S}$ exhibits finite nondeterminism, then the algorithm in Fig. 10 uses only a bounded number of registers in addition to the objects used by* **A**.

Proof. By inspection, besides the shared objects used in **A**, all the additional nonauxiliary shared objects used by the algorithm in Fig. 10 are registers. Since $\mathcal{S}$ contains type register, it follows that the algorithm uses only objects of types in $\mathcal{S}$.

We now show that the algorithm in Fig. 10 solves one-resilient Consensus for the three processes in $\mathcal{Q}$. To this end, in the rest the proof we fix an arbitrary execution of the algorithm in Fig. 10 in which at most one process in $\mathcal{Q}$ crashes. Henceforth, our discussion refers to this fixed execution. We shall prove that in this execution the three properties of Consensus – Termination, Validity and Agreement – are satisfied.

First we make some definitions. Let S_j be the value of S when S contains a schedule of exactly j steps (undefined if S never has j steps). Let W_j be the value assigned to W when S_j is assigned to S (both auxiliary variables are updated in the same "atomic" action in lines 16–17.) Note that if $W_j[i,r] \neq \perp$, then for all $j' \geq j$ where $W_{j'}$ is defined, $W_{j'}[i,r] = W_j[i,r]$. This is because at most one process in $\mathcal{Q}$ wins $TAS[i,r]$ (Lemma 19(d)), and that is the only process that may assign its id to $W[i,r]$ (line 16). Let $\mathcal{I}_j$ be the set of initial configurations I of **A** that satisfy the following property: If $W_j[i,1] = k \neq \perp$, then the initial value of P_i in I is equal to the initial value of Q_k. That is, all the initial configurations in $\mathcal{I}_j$ agree on exactly the initial values of all processes in $\mathcal{P}$ that have at least one step in S_j; the initial value of any such process, say P_i, is equal to the initial value of the process in $\mathcal{Q}$ that simulated P_i's first step in S_j (i.e., the process Q_k such that $W_j[i,1] = k$). Note that if $j \leq j'$, then $\mathcal{I}_j \supseteq \mathcal{I}_{j'}$.

We now define S^* and W^*. The intuition behind these definitions is that S^* and W^* are the final values of the auxiliary variables S and W, if these variables eventually stop being updated. If S and W keep changing forever, S^* and W^* are the "limit" values of these variables. More precisely, S^* is defined as follows: if S_j is defined then the jth step of S^* is the last step of S_j; otherwise, S^* does not have a jth step. W^* is defined as follows:

$$W^*[i,r] = \begin{cases} \perp & \text{if for every } j,\ W_j[i,r] = \perp, \\ k & \text{if for some } j,\ W_j[i,r] = k \neq \perp. \end{cases}$$

Note that W^* is well-defined because if $W_j[i,r] = k$ then, for any $j' \geq j$ such that $W_{j'}$ is defined, $W_{j'}[i,r] = k$ as well. Finally, we define $\mathcal{I}^*$ (derived from W^*, just as $\mathcal{I}_j$ is derived from W_j) as the set of initial configurations I of **A** that satisfy the following property: If $W^*[i,1] = k \neq \perp$, then the initial value of P_i in I is equal to that of Q_k.

With these definitions, it is easy to show that the following properties are invariants of the algorithm in Fig. 10: For all $j \geq 0$ such that S_j is defined,

(a) S_j is a schedule of **A** that is applicable to every $I \in \mathcal{I}_j$.
(b) $W_j[i,r] \neq \perp$ if and only if S_j has at least r steps of P_i.
(c) For any process P_i, the suffix of S_j consisting of all steps after P_i has decided contains no step of P_i.
(d) Let P_i be the process such that the last step of S_j is a step of P_i, and σ be the value assigned to $State[i]$ just after S_j is assigned to S. For every $j' \geq j$, if $S_{j'}$ is defined and contains no steps of P_i after its prefix S_j, then σ is the state of P_i in configuration $S_{j'}(I)$ of **A**, for all $I \in \mathcal{I}_{j'}$.
(e) If some process has r steps in S_j, then at most one process has fewer than $r-1$ steps in S_j.

We are now ready to prove that the fixed execution satisfies the three properties of Consensus. For each $i \in \{0, \ldots, n-1\}$ and $r \geq 1$, let $E_{i,r}$ denote the subexecution consisting of steps taken only by the processes in $\mathcal{Q}$ when they apply TEST&SET operations to $TAS[i,r]$ in line 7.

Termination: Assume, for contradiction, that some correct process never decides. Then, that process must always find $Decision = \perp$ in line 2 and line 8. Therefore, no process (correct or crashed) ever executes line 20 – because once $Decision$ is set to a non-$\perp$ value, it is never set back to $\perp$. Hence,

$$Decision = \perp, \text{ forever.} \qquad (*)$$

Claim 20.1. At least one correct process in $\mathcal{Q}$ executes the **while** loop (lines 3–23) infinitely often.

Proof of Claim 20.1. Suppose, for contradiction, that no correct process executes the **while** loop (lines 3–23) infinitely often. Since at most one process in $\mathcal{Q}$ may crash, there are at least two correct processes, say Q and Q'. By assumption, neither Q nor Q' executes the **while** loop infinitely often. By line 2 of Fig. 10 and $(*)$, these two processes cannot exit the **while** loop. Therefore, eventually each of Q and Q' applies a nonterminating **coexecute** statement (lines 6–9). Let i and r be the values of Q's local variables id and rd after Q has started its nonterminating **coexecute** statement; let i' and r' be values with corresponding interpretations for Q'. By line 7, it must be that Q and Q' apply a nonterminating TEST&SET operation to $TAS[i,r]$ and $TAS[i',r']$, respectively. In other words, Q and Q' are blocked in $E_{i,r}$ and $E_{i',r'}$, respectively. Let Q'' be the remaining process (other than Q and Q') in $\mathcal{Q}$. By Lemma 19(b), Q'' is potentially interrupting in both

$E_{i,r}$ and $E_{i',r'}$; therefore, $(i,r) = (i',r')$. By Lemma 19(a), at most one process can be blocked in $E_{i,r}$, contradicting that both Q and Q' are. □ Claim 20.1

Claim 20.2. For any $i \in \{0,1,\ldots,n-1\}$, $Instr[i]$ is nondecreasing.

Proof of Claim 20.2. The claim follows immediately from the following two facts. For any positive integer r,

(i) at most one process writes $r+1$ into $Instr[i]$; and
(ii) $Instr[i]$ is set to r before it is set to $r+1$.

Fact (i) is true because, to write $r+1$ into $Instr[i]$ (line 22), a process must win $TAS[i,r]$ (lines 7 and 10), and at most one process can do so (by Lemma 19(d)). Fact (ii) is true because, before a process writes $r+1$ into $Instr[i]$ (line 22), it must have previously read r from $Instr[i]$ (line 3). □ Claim 20.2

Since $Instr[i]$ is nondecreasing, it follows that, over time, its value is either unbounded or it attains some maximum value and never changes thereafter.

Claim 20.3. For any $i \in \{0,1,\ldots,n-1\}$, if $Instr[i]$ is bounded with maximum value r, then some process in $\mathcal{Q}$ crashes either (a) during its first invocation of TEST&SET on $TAS[i,r]$ or (b) after winning $TAS[i,r]$ in line 7 but before incrementing $Instr[i]$ in line 22.

Proof of Claim 20.3. Suppose, for contradiction, that there is some $i \in \{0,1,\ldots,n-1\}$ and some $r > 0$ such that $Instr[i]$ is bounded with maximum value r, and no process in $\mathcal{Q}$ crashes (a) during its first invocation of TEST&SET on $TAS[i,r]$, or (b) after winning $TAS[i,r]$ in line 7 but before incrementing $Instr[i]$ in line 22. Without loss of generality, let r be the minimum value so that these conditions are met.

By Claim 20.1, some correct process, say Q_k, executes the **while** loop (lines 2–23) infinitely often. Since in each iteration Q_k increments its local variable id by one modulo n (line 23), Q_k executes the while loop with $id = i$ infinitely often. By Claim 20.2 and our hypothesis, after some point in time, $rd = r$ forever. Therefore, Q_k executes the while loop with $id = i$ and $rd = r$ infinitely often. By the minimality of r, for each $t \in \{0,1,\ldots,n-1\}$, one of the following is the case: (i) $Instr[t]$ is unbounded; or (ii) $Instr[t]$ is bounded and its maximum value is at least r; or (iii) some process in $\mathcal{Q}$ crashes while its local variable $id = t$. Since at most one process in $\mathcal{Q}$ may crash, eventually for all but at most one $t \in \{0,1,\ldots,n-1\}$, $Instr[t] \geq r$. Therefore, there is a time after which in each of its infinitely many executions of the while loop with $id = i$ and $rd = r$, Q_k will find the conditional in line 4 to be true. Thus, Q_k invokes infinitely many TEST&SET operations on $TAS[i,r]$ (line 7). Hence, $E_{i,r}$ is nonempty. Since, by assumption (a), no process in $\mathcal{Q}$ crashes during its first invocation of a TEST&SET on $TAS[i,r]$, there is no potentially interrupting process in $E_{i,r}$. By Lemma 19(c), some process wins $TAS[i,r]$. By assumption (b), this process does not crash before incrementing $Instr[i]$ to $r+1$ in line 22. This contradicts the assumption that r is the maximum value that $Instr[i]$ attains. □ Claim 20.3

By Claim 20.3, there is at most one element $i \in \{0,1,\ldots,n-1\}$ such that $Instr[i]$ is bounded. (Otherwise, there would have to be two processes in $\mathcal{Q}$ that crash.) If there is such an element, let $\hat{\imath}$ be it – otherwise, let $\hat{\imath}$ be an arbitrary element of $\{0,1,\ldots,n-1\}$. Thus, for all $i \in \{0,\ldots,n-1\} \setminus \{\hat{\imath}\}$ the value of $Instr[i]$ is unbounded. Before incrementing $Instr[i]$ to $r+1$ (in line 22), a process in $\mathcal{Q}$ must have previously set $W[i,r]$ to a non-$\perp$ value (line 16). Thus, for all $i \in \{0,\ldots,n-1\} \setminus \{\hat{\imath}\}$ and all $r \geq 1$, $W^*[i,r] \neq \perp$. By Invariant (b), S^* contains infinitely many steps of P_i, for all $i \in \{0,\ldots,n-1\} \setminus \{\hat{\imath}\}$. This means that S^* is an infinite schedule in which at most one process in $\mathcal{P}$ crashes. By Invariant (a), S^* is a schedule of $\mathbf{A}$ that is applicable to every $I \in \mathcal{I}^*$. By $(*)$, for every $I \in \mathcal{I}^*$ and $i \neq \hat{\imath}$, there is no prefix S' of S^* such that P_i has decided in $S'(I)$. This contradicts the fact that $\mathbf{A}$ is a one-resilient Consensus algorithm for n processes (in particular, it contradicts the Termination property of $\mathbf{A}$).

We reached this contradiction by assuming that the Termination property in violated. Thus, Termination is satisfied.

Agreement and Validity: By the Termination property shown above, S^* is a finite schedule.

Claim 20.4. For any process $Q \in \mathcal{Q}$ and value $v \in \{0,1\}$, if Q writes v into $Decision$ (line 20), then some process in $\mathcal{P}$ decides v in $S^*(I)$, for every $I \in \mathcal{I}^*$.

Proof of Claim 20.4. Let i and r be the values of its local variables id and rd when Q writes v into $Decision$ in line 20. Let σ be the value that Q last wrote into $State[i]$ in line 18 before Q assigns v to $Decision$. In other words, Q simulates the rth step of P_i; as a result of this step P_i enters state σ in which it decides v (cf. line 19).

By Invariant (c), P_i has no additional steps in S^* after it decides. By Invariant (d), σ is the state of P_i in $S^*(I)$, for every $I \in \mathcal{I}^*$. Since Q assigns v to $Decision$ in line 20, it must be that P_i decides v in σ. Thus, P_i decides v in $S^*(I)$, for every $I \in \mathcal{I}^*$ – as wanted. □ Claim 20.4

To prove that Agreement is satisfied, suppose that two distinct processes in $\mathcal{Q}$ decide the values v and v', respectively, that they read from $Decision$. Thus, both v and v' are written into $Decision$ by some processes in $\mathcal{Q}$ in line 20. By Claim 20.4, some process in $\mathcal{P}$ decides v in $S^*(I)$ and some process in $\mathcal{P}$ decides v' in $S^*(I)$, for every $I \in \mathcal{I}^*$. By the Agreement property of $\mathbf{A}$, $v = v'$. Therefore, the algorithm in Fig. 10 satisfies Agreement.

Finally we prove that Validity is satisfied. This is obvious if both 0 and 1 are initial values of the processes in $\mathcal{Q}$. Thus, we may assume that, for some $v \in \{0,1\}$, all processes in $\mathcal{Q}$ have initial value v. Let I_v be the initial configuration of $\mathbf{A}$ in which every process in $\mathcal{P}$ has initial value v. Clearly, $I_v \in \mathcal{I}^*$. Suppose now that some process in $\mathcal{Q}$ decides the value u that it reads from $Decision$. By Claim 20.4, some process P in $\mathcal{P}$ decides u in $S^*(I)$, for every $I \in \mathcal{I}^*$. In particular, P decides u in $S^*(I_v)$. By the Validity property of $\mathbf{A}$, $u = v$. Thus, if all processes in $\mathcal{Q}$ have initial value v, then any process that decides, must decide v. Therefore, the algorithm in Fig. 10 satisfies Validity.

Bounded number of additional registers: To complete the proof of Lemma 20, it remains to show that if every type in $\mathcal{S}$ exhibits finite nondeterminism, then the algorithm in Fig. 10 uses only a bounded number of registers in addition to the

objects used by $\mathbf{A}$. To show this, it suffices to prove that there is a bound on the number of the test-and-set-register objects in array TAS used by executions of the algorithm in Fig. 10 in which at most one process crashes. Finally, to prove this it suffices to prove that the length of the schedule in the auxiliary variable S is bounded.

To prove this we proceed as follows. For any initial configuration I of algorithm $\mathbf{A}$, we define a tree T_I, whose nodes are schedules S that satisfy the following three properties:

(1) S is a finite schedule of $\mathbf{A}$ that is applicable to I.
(2) S does not contain a step of P_i after P_i has decided.
(3) If S has r steps of some process, then at most one process has fewer than $r-1$ steps in S.

The tree T_I has an edge from node (schedule) S to S' if and only if S is a prefix of S' and $|S'| = |S| + 1$.

Claim 20.5. For any initial configuration I of $\mathbf{A}$, T_I is finite.

Proof of Claim 20.5. Suppose, for contradiction, that T_I is infinite, for some I. Since every type in $\mathcal{S}$ exhibits finite nondeterminism, every node in T_I has finite degree. By König's Lemma, T_I has an infinite path. It is easy to see that this path corresponds to an infinite schedule S^∞ such that: (i) S^∞ is applicable to I (by property (1) of the nodes of T_I); (ii) some correct process never decides in S^∞ (by property (2) of the nodes of T_I); and (iii) at most one process is faulty in S^∞ (by property (3) of the nodes of T_I). These three facts contradict that $\mathbf{A}$ is a one-resilient Consensus algorithm for $\{P_0, P_1, \ldots, P_{n-1}\}$. $\square$ Claim 20.5

By Claim 20.5, there is a positive integer ℓ_I that is an upper bound on the length of any schedule S that satisfies properties (1), (2) and (3). Let

$$\ell = \max\{\, \ell_I : \ I \text{ is an initial configuration of } \mathbf{A} \,\}$$

(such a maximum exists, since $\mathbf{A}$ has finitely many initial configurations). By Invariants (a), (c) and (e), the schedule stored in the auxiliary variable S in any execution of the algorithm in Fig. 10 in which at most one process crashes satisfies properties (1), (2) and (3). Therefore, the length of the schedule stored in the auxiliary variable S in any execution of the algorithm in Fig. 10 in which at most one process crashes is bounded by ℓ, as wanted. $\square$

Lemma 20 immediately implies:

Theorem 3 *Let $\mathcal{S}$ be any set of object types that includes* register*, and $n \geq 4$ be any integer.*

(a) If $\mathcal{S}$ implements one-resilient Consensus among n processes, then $\mathcal{S}$ implements one-resilient Consensus among three (and therefore among $n-1$) processes.
(b) If $\mathcal{S}$ boundedly implements one-resilient Consensus among n processes and every type in $\mathcal{S}$ exhibits finite nondeterminism, then $\mathcal{S}$ boundedly implements one-resilient Consensus among three (and therefore among $n-1$) processes.

Part (b) of Theorem 3, regarding *bounded* implementability, applies if $\mathcal{S}$ contains only types that exhibit finite nondeterminism. It is natural to inquire whether this requirement is necessary. In fact, it is not: the result holds for *all* sets of types, even those that contain types with infinite nondeterminism. The proof of this stronger result is based on a more complicated simulation that uses $O(n)$ "resettable" test-and-set-register objects and a garbage collection technique to recycle them. This simulation is described in [11].

5 Conclusion

The results of this paper, together with those in [4], completely characterise the relationship between the solvability of Consensus among different numbers of processes. More precisely, the results show that, given any set $\mathcal{S}$ of object types and any integers n, t such that $n-1 > t \geq 1$, the statement

> t-resilient Consensus among n processes is solvable using $\mathcal{S}$ if and only if
> t-resilient Consensus among $n-1$ processes is solvable using $\mathcal{S}$

is valid if and only if $t \geq 2$ or $n \geq 4$. Besides this characterisation, these results also reveal a qualitative difference between level one and other levels of the Consensus hierarchy [6,7]. If $\mathcal{S}$ contains an object type at level two or above of the Consensus hierarchy, the result of Chandra et al. implies that the statement above is valid. On the other hand, if $\mathcal{S}$ contains only object types at level one of the Consensus hierarchy, our result in Sect. 3 shows that the statement is false. This shows that level one of the Consensus hierarchy may have a richer structure than other levels.

Type dor defined in Sect. 3 is the only deterministic type we know of that is at level one of the Consensus hierarchy and has no one-resilient implementation for three or more processes using only registers.

Simulations similar to that described in Sect. 4.2 were used first by Borowsky and Gafni [1] (see also [13,2]), and later by Chandra et al. [4]. The main innovation here is the TAS register implementation of Sect. 4.1. Borowsky and Gafni used a simulation that requires processes to agree on the outcome of each step by solving (a restricted form of) Consensus using only (read/write) registers. Instead of agreeing on the outcome of a step, we use (a similarly restricted form of) TAS registers to ensure that only one process simulates each step. Unlike the simulation of Borowski and Gafni, which applies only when the simulated algorithm uses (read/write) registers (or types of equivalent power), our simulation does not place any restriction on the types of the objects used by the simulated algorithm: these can of any type(s) whatseover. The simulation of Chandra et al. is also general in the sense that it works regardless of the types of the objects used by the simulated algorithm, but it applies to a context in which TAS registers are available directly and need not be implemented.

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