Problem 7.1 [Van Emde Boas in Space]. Recall the “saving space” version of the van Emde Boas predecessor data structure described in Lecture 11:

A van Emde Boas structure $V$ of universe size $u$ consists of:

- $V\.min =$ the minimum element in $V$, not stored recursively
- $V\.max =$ the maximum element in $V$, stored recursively
- $V\.summary =$ a van Emde Boas structure of universe size $\sqrt{u}$, representing which clusters are nonempty (excluding $V\.min$)
- $V\.cluster =$ a hash table mapping a cluster number in \{0, 1, \ldots, \sqrt{u} - 1\} to a van Emde Boas structure of universe size $\sqrt{u}$ representing the elements in that cluster, but only for clusters that are nonempty (excluding $V\.min$)

(a) Prove matching (up to constant factors) upper and lower bounds on the worst-case space occupied by a “saving space” van Emde Boas structure of universe size $u$ storing $n$ elements. (Count the number of words of space, as an asymptotic function of $n$ and $u$, ignoring constant factors.) As long as your upper and lower bounds match up to constant factors, we will accept weak lower bounds of the form “for any sufficiently large $n$, there is a $u$ such that the space is at least $f(n, u)$”. Equivalently, it suffices to produce an infinite family of instances with arbitrarily large $n$; you can assume a relation between $n$ and $u$, as long as $n$ can become arbitrarily large. On the other hand, your upper bound must work for all $n$ and all $u$.

(b) How can you modify the data structure to achieve $O(n)$ words of space?

\[^{1}\text{http://courses.csail.mit.edu/6.851/spring21/lectures/L11.html?notes=7} \]