**Problem 4.1 [Generically Speaking]**. Show that any point set \( S \) can be transformed into a generic point set \( S' \) preserving the strict ordering of coordinates and approximately preserving \( \text{OPT} \).

**Solution**: It suffices to solve the problem one dimension at a time:

**Lemma 1.** For any point set \( S \), there exists a point set \( S' \) such that

(a) No two points in \( S' \) share an \( x \) coordinate.

(b) Strict relative ordering of points is preserved.

(c) \( |\text{OPT}(S')| = O(|\text{OPT}(S)|) \).

**Proof.** Let \( T = \text{OPT}(S) \) and, for any \( x \), let \( S_x = [(x, y_1), (x, y_2), \ldots, (x, y_m)] \subseteq S \) denote the list of all points in \( S \) with that \( x \) coordinate, sorted in increasing order of \( y \).

Let \( \Delta x > 0 \) be smaller than any positive difference between \( x \) coordinates of points in \( S \). Then it is also smaller than any positive difference between \( x \) coordinates of points in \( T \). Let \( \varepsilon = \Delta x/n \).

Now we construct two sets \( S' \subseteq T' \):

(a) For each \((x, y_i) \in S_x\), add \((x + i\varepsilon, y_i)\) to \( S' \) and \( T' \).

(b) For each \((x, y_{i+1}) \in S_x\), add \((x + i\varepsilon, y_{i+1})\) to \( T' \).

(c) For each \((x, y) \in T\), add \((x, y)\) and \((x + n\varepsilon, y)\) to \( T' \).

Informally, for each vertical segment \( S_x \) we add a diagonal line going up and to the right to \( S' \) and a backwards "N" to \( T' \).

![Diagram](image)

Figure 1: "*" denotes a point in both \( S' \) and \( T' \), while "X" denotes a point in \( S' \) only.

It is easy to see that \( T' \) is satisfied, and that \( |\text{OPT}(S')| \leq |T'| \leq 4|T| = 4|\text{OPT}(S)| \).  

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1 Not to be confused with a forwards “И”, which has a thinner diagonal stroke and more symmetrical serifs.