6.851: Advanced Data Structures, Spring 2021

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Problem Set 3 Solutions

Due: Thursday, March 11, 2021

Problem 3.1 [Thinking Outside The Box]. The orthogonal range query data structure described in Lecture 4 supports the following operation:

> ***inside*** $(\mathbf{p}, \mathbf{q})$: return all k points *inside* the bounding box spanned by points $\mathbf{p}$ and $\mathbf{q}$.

Implement the following operation using $O(d)$ calls to inside():

> ***outside*** $(\mathbf{p}, \mathbf{q})$: return all k points *outside* the bounding box spanned by points $\mathbf{p}$ and $\mathbf{q}$.

Other than the black-box calls to inside(), the running time of your algorithm should be $O(d+k)$.

Solution: The idea is to subdivide the space outside the query range into $O(d)$ disjoint ranges, query each of them, and combine the results.

Without loss of generality, $p_i < q_i$ for all i.

For each dimension, we can partition the space into 3 parts: $L_i = \{x \in \mathbb{R}^d \mid x_i < p_i\}$, $C_i = \{x \in \mathbb{R}^d \mid p_i \le x_i \le q_i\}$, and $R_i = \{x \in \mathbb{R}^d \mid x_i > q_i\}$. Then we can write the exterior as

$$\begin{aligned}
\mathbb{R}^d \setminus (C_1 \cap \cdots \cap C_d) = {} & L_1 \sqcup R_1 \\
& \sqcup (C_1 \cap L_2) \sqcup (C_1 \cap R_2) \\
& \sqcup (C_1 \cap C_2 \cap L_3) \sqcup (C_1 \cap C_2 \cap R_3) \\
& \vdots \\
& \sqcup (C_1 \cap \cdots \cap C_{d-1} \cap L_d) \sqcup (C_1 \cap \cdots \cap C_{d-1} \cap R_d).
\end{aligned}$$

Each of these $2d$ terms is an orthogonal range. For example, $C_1 \cap L_2 = [p_1, q_1] \times (-\infty, p_2) \times \mathbb{R}^{d-2}$. We call inside() on each these ranges in order and concatenate the results.

Creating the first range takes $O(d)$ time. Each successive range differs from the previous range in only $O(1)$ coordinates, so we can update it in $O(1)$ time. Concatenating the results takes $O(k)$ time. So the total running time outside the $2d$ calls to inside() is $O(d+k)$.

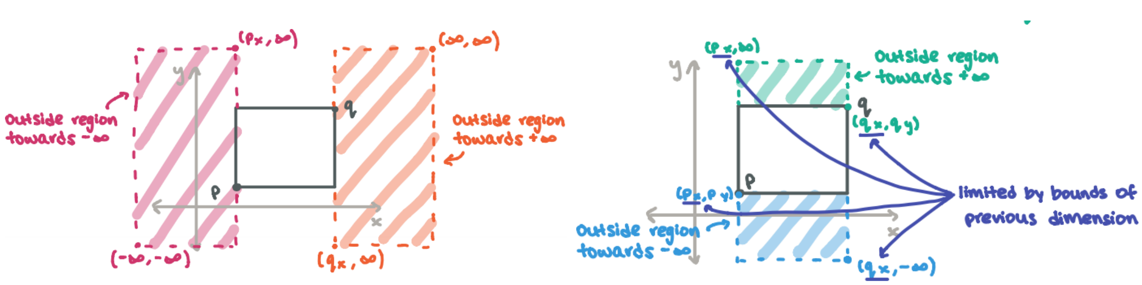

Figure 1: $d = 2$ example. [Figure by Shana Mathew]