Today: Dynamic Optimality
- geometry → BSTs
  - online
  - offline

Summary:

Lower bounds:

Signed Greedy

Wilber 1(p) Wilber 2

Upper bounds:

- Splay
- Greedy
- Tango

\( \Theta(\text{max IRB}) \)

independent rectangle bounds

\{ conjectured \( O(1) \)-competitive

- proved \( O(lg lg n) \)-competitive
Geometry \rightarrow \text{offline BST:}
- \text{treap: there's a unique tree that's a BST w.r.t. key \& a heap w.r.t. priority}
  - root = \text{min priority item}
  - BST property splits remainder
  - recurse on left \& right subtrees
- \text{here: key = key, priority = next touch time}
- priorities aren't unique (>1 touch/time) so neither is treap
  \rightarrow \text{like B-tree}

- \text{multitreap: there's a unique tree that's search tree w.r.t. key \& strict heap w.r.t. priority}
  \begin{align*}
  \rightarrow & \; x < y \\
  \Rightarrow & \; x = y
  \end{align*}
  \text{(strict increase) (only equal priorities in a single node)}
- root consists of all min priority items
- search tree property splits remainder
- recurse

- effectively we maintain the unique multitreap
- really maintain "disambiguates" a treap that the multitreap
Example:
next touch time
1 2 3 4 5 6
touch
next multi-tree

ambiguous - OK as is
Geometry online BST:
- maintain multitreap like right column but with more ambiguity
- initially 1 big ambiguous node
- when touching a node $x$:
  split ambiguous node into $<x, x, >x$
- to re-organize subtree of touched nodes: merge into one ambiguous root node
  (note all just touched $\Rightarrow$ not ambiguous)
- same cost if we have:

Split trees: (ambiguous nodes)
- $\text{make-tree}(x_1, x_2, \ldots, x_k)$ costs $O(k)$
- $T, \text{split}(x) \rightarrow (T_{<x}, x, T_{>x})$ costs $O(1)$ amortized

Details:
- $O(\min \{ \lg |T_{<x}|, \lg |T_{>x}| \})$ split suffices via potential
  $\Phi = \sum_{\text{split tree } T} (|T| - \lg |T|)$
- B-trees with min & max augmentation
- simulate by BST (interleaving min/max)
- combine multiple split trees into BST similar to Tango trees (marked nodes)
Example:
1 2 3 4 5 6

Initial multitreap:

1 2 3 4 5 6

↑ split

↓

1 2

↓

3

4 5 6

↑

Collapse

1

2

3

4 5 6

→

2

4 5 6

→

1 3

↑

↑

↑

↑

4 5

↑

1 2 3 6

→

1 2 3 6

→

1 3

↑

↑

↑

↑

4 5

↑

1 2 6

→

1 2

↑

↑

↑
Problem 1: Show that assuming accesses $X$ are all to distinct keys ($x$ coords.) affects $|\text{OPT}(X)|$ by at most $O(|X|)$. 

i.e. transform $X \rightarrow X'$ by spreading out identical keys (but otherwise preserving order) such that $|\text{OPT}(X')| \leq |\text{OPT}(X)| + O(|X|)$

(3 suffices)

Problem 2: Prove a logarithmic separation between the BST & pointer-machine models (for successful searches) 

i.e. construct an infinite family of access sequences $x_1, x_2, \ldots, x_m \in \{1, 2, \ldots, n/3\}$ such that 

$\text{OPT}_{\text{BST}}(X) = \Theta(m \lg n)$ 

and $\text{OPT}_{\text{PM}}(X) = \Theta(m)$

Discuss relationship between open problems, projects, papers, and authorship