6.851	Lecture 18	April 13, 2010

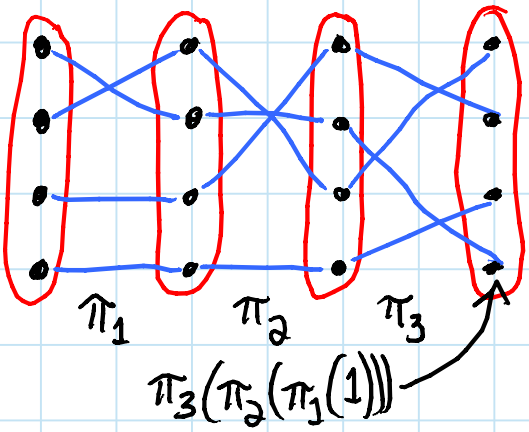
Dynamic connectivity lower bound:

[Pătraşcu & Demaine – SICOMP 2006]

inserting/deleting edges & connectivity queries require $\Omega(\lg n)$ cell probes/op.
even if connected components are paths
even amortized (but here prove for worst case)

Proof:

- consider $\sqrt{n} \times \sqrt{n}$ grid with perfect matching between columns i & $i+1$ for each i, forming permutation π_i



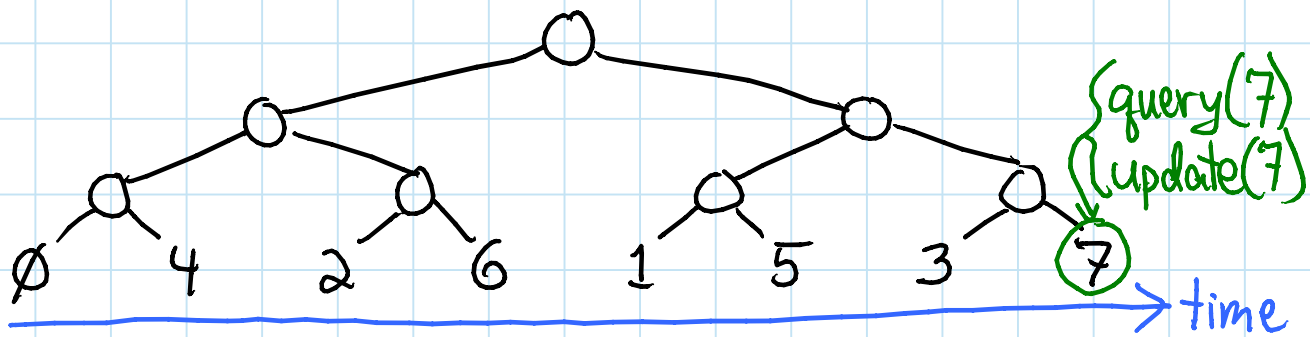
- block operations:
  - update(i, π): $\pi_i \leftarrow \pi$
    $= O(\sqrt{n})$ edge deletions & insertions
  - verify-sum(i, π): $\sum_{j=1}^{i} \pi_j = \pi$?
    compose
    $= O(\sqrt{n})$ connectivity queries
- Claim: $\sqrt{n}$ updates + $\sqrt{n}$ verify sums require $\Omega(\sqrt{n} \cdot \sqrt{n} \cdot \lg n)$ cell probes
  $\Rightarrow \Omega(\lg n)$/op.

Bad access sequence:

- for i in bit-reversal sequence:
  - verify-sum$(i, \sum_{j=1}^{i} \pi_j)$ $\Rightarrow$ answer = yes (but DS must check)
  - update$(i, \underbrace{\pi_{random}}_{\text{uniform random permutation}})$

000	000	0
001	100	4
010	010	2
011	110	6
⋮	⋮	⋮

- build tree over time:



- left & right subtrees of each node <u>interleave</u>

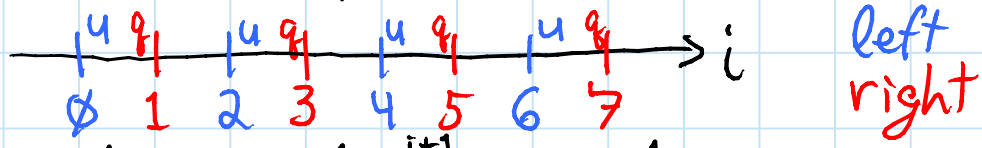
- <u>Claim</u>: for every node v in tree, say with ℓ leaves in its subtree, during right subtree of v (time interval) must do $\Omega(\ell\sqrt{n})$ expected cell probes reading cells last written during left subtree

- sum lower bound over all nodes:
  - read r of write w only counted at $\mathrm{lca}(r, w)$
  - linearity of expectation

$\Rightarrow \Omega(n \lg n)$ lower bound total
(each leaf in $\Theta(\lg n)$ subtrees)

Proof of claim:

- left subtree has $\ell/2$ updates with $\ell/2$ rand. perms.
- any encoding of these permutations must use $\Omega(\ell\sqrt{n}\lg n)$ bits [information/Kolmogorov theory]
- if claim fails, find smaller encoding $\Rightarrow$ contradict.
- setup: know the past (before v's subtree)
- goal: encode (verified) sums in right subtree
  $\Rightarrow$ can recover (updated) perms. in left subtree



$$\pi_i = \underbrace{\pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}}_{\text{farther left} \Rightarrow \text{known}} \circ \underbrace{\sum_{j=1}^{i+1} \pi_j}_{\text{right } q} \circ \underbrace{\pi_{i+1}^{-1}}_{\text{not yet updated}}$$

Warmup: query is sum(i) $\rightarrow \sum_{j=1}^{i} \pi_j$ (partial sums)

- let $R = \{$cells read during right subtree$\}$
  $W = \{$cells written during left subtree$\}$
- encode $R \cap W$ (address & contents of each cell)

$\Rightarrow |R \cap W| \cdot O(\lg n)$ bits [assume poly. space $\Rightarrow w = \Theta(\lg n)$]

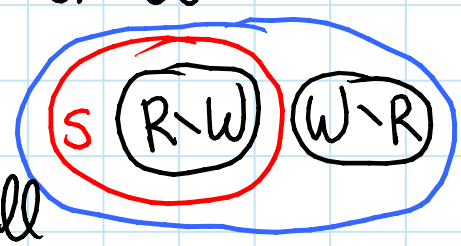
- decoding alg. for sums in right subtree:
  - simulate sum queries in right subtree
  - to read cell written in right subtree: easy
    in left subtree: $R \cap W$
    in past: known

$\Rightarrow |R \cap W| \cdot \cancel{O(\lg n)} = \Omega(\ell\sqrt{n}\,\cancel{\lg n})$

$\Rightarrow |R \cap W| = \Omega(\ell\sqrt{n})$ ✓

Verify-sum instead of sum:

- permutations π given to verify-sum encode the information we want ($\Rightarrow$ no info. LB)
- setup:
  - know (fixed) past
  - don't know updates in left subtree
  - don't know queries in right subtree
  - but know that queries return YES
- decoding idea:
  - simulate all possible input permutations for each query in right subtree
  - know one returns YES, all others NO
- trouble: incorrect query simulation reads cells $R' \neq R$
  - if read $r \in R' \setminus R$, it must be incorrect
  - but can't tell whether $r \in W \setminus R$ or $\text{past} \setminus (R \cap W)$
  - can't afford to encode R or W
- idea: encode separator S for $R \setminus W$ & $W \setminus R$



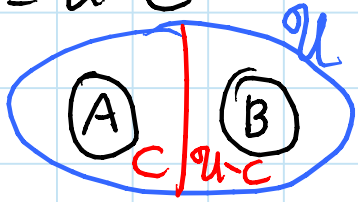
- when decoding, to read cell written in right subtree: easy
  - in $R \cap W$: encoded explicitly
  - in S: must be in past $\Rightarrow$ known
  - not in S: must not be in $R \Rightarrow$ incorrect: ABORT
- only one simulation returns YES: rest NO or ABORT

$\Rightarrow$ recover desired permutation

$\Rightarrow |\text{encoding}| = \Omega(\ell\sqrt{n}\lg n)$

Separators:

- given universe $\mathcal{U}$ & number m
- separator family $\mathcal{S}$ for size-m sets if $\forall A, B \subseteq \mathcal{U}$ with $|A|, |B| \le m$ & $A \cap B = \emptyset$: $\exists C \in \mathcal{S}$ such that $A \subseteq C$ & $B \subseteq \mathcal{U} \setminus C$
- claim: $\exists$ separator family $\mathcal{S}$ with $|\mathcal{S}| \le 2^{O(m + \lg\lg|\mathcal{U}|)}$



- proof sketch:
  - perfect hash family $\mathcal{H}$ with $|\mathcal{H}| \le 2^{O(m + \lg\lg|\mathcal{U}|)}$
    [Hagerup & Tholey – STACS 2001]
    gives mapping from A & B to $O(m)$-size table
  - store A-or-B bit in each table entry
  - $2^{O(m)}$ such vectors

$\Rightarrow 2^{O(m)} \cdot 2^{O(m + \lg\lg|\mathcal{U}|)} = 2^{O(m + \lg\lg|\mathcal{U}|)}$

Encoding: $R \cap W$ + separator of $R \setminus W$ & $W \setminus R$

- size: $|R \cap W| \cdot O(\lg n) + O(|R| + |W| + \lg\lg n)$
  $= \Omega(\ell\sqrt{n} \lg n)$

$\Rightarrow |R \cap W| = \Omega(\ell\sqrt{n})$ $\Rightarrow$ claim

or $|R| + |W| = \Omega(\ell\sqrt{n} \lg n)$ $\Rightarrow \Omega(\lg n)$ for op.

$+\lg\lg n$

$\square$

Update-query trade-off: (possible by same technique)

$$t_q \lg \frac{t_u}{t_q} = \Omega(\lg n) \quad \& \quad t_u \lg \frac{t_q}{t_u} = \Omega(\lg n)$$

- for $t_u = \Omega(t_q)$, trees can match
  (small mods. to link-cut trees)
- for $t_u = \Omega(\lg n \, (\lg \lg n)^3)$, can match
  [Thorup–STOC 2000]