On Weak $d$-universal Hash Families. Recall that a set $\mathcal{H}$ of hash functions is a weak $d$-universal family if, for all $x, y \in U$ with $x \neq y$,

$$\Pr_{h \sim \mathcal{H}} \left\{ h(x) = h(y) \right\} = \frac{d}{m}.$$ 

Let $U = \mathbb{Z}_2^\ell$ (the set of bit vectors of length $\ell$). For a given $k \times \ell$ binary matrix $M$, we define a hash function $h_M : U \rightarrow \mathbb{Z}_2^k$ as $h_M(x) = M \cdot x$, where additions and multiplications are done modulo 2. Show that the family $\mathcal{H} = \{ h_M \mid M \text{ is a binary } k \times \ell \text{ matrix} \}$ is weakly 1-universal.

Deterministic $y$-fast tries. Suppose you have a dynamic perfect hash function $h$ such that:

- $h$ is constructible in deterministic linear time;
- $h(x)$ can be evaluated in $O(1)$ worst case, deterministic time;
- insertions and deletions take $O(\lg^5 u)$ worst case time;

Use $h$ to modify the $y$-fast trie data structure to support insertion, deletion, predecessor, and successor in $O(\lg \lg u)$ amortized deterministic (rather than randomized) time.

Range Existence Queries. Given a set $S$ of integers, the range existence query $\text{req}(a, b)$ asks whether there is any element in $S \cap \{a, a + 1, \ldots, b\}$. Suggest an $O(n \lg u)$-space data structure that stores a static set $S$ of $n$ integers from $U = \{0, 1, \ldots, u - 1\}$ and answers range existence queries in expected $O(1)$ time.

Hint: Think why LCA queries in a perfect binary tree are easy.