Lecture 20

Ordered file maintenance [Itai, Konheim, Rodeh – ICALP 1981; Bender, Demaine, Farach-Colton – FOCS 2000]

Goal: store N elements in specified order in array of size $O(N)$ with gaps of size $O(1)$

$\Rightarrow$ scanning K consecutive elements costs $O(\lceil \frac{K}{B} \rceil)$ m.t.

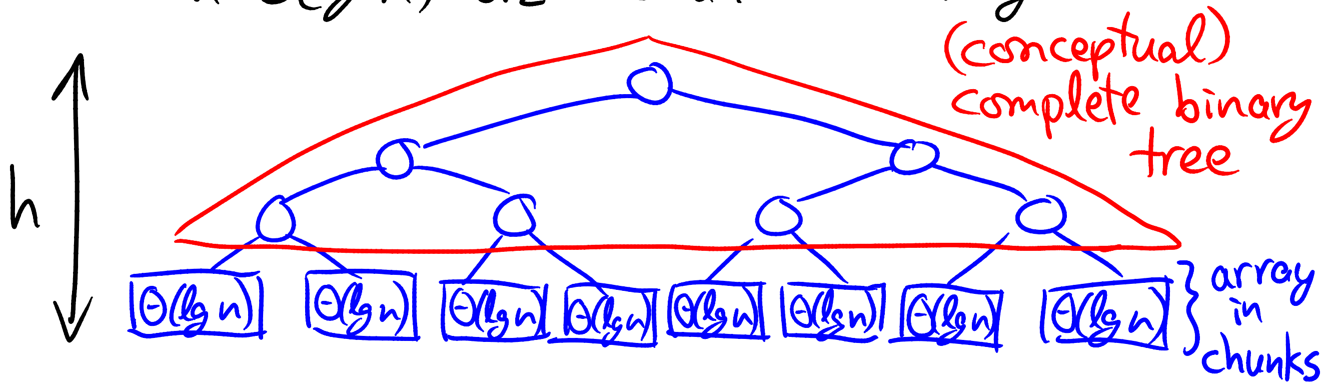
subject to element deletion/insertion between two given elements by re-arranging elements in array interval of $O(\lg^2 N)$ amortized

$\Rightarrow$ costs $O(\frac{\lg^2 N}{B})$ amortized memory transfers

Rough idea: upon updating element, ensure locally not too dense/sparse

- grow an interval around the element until not "too" dense or sparse ~ ratio $\frac{\text{elements}}{\text{space}}$
- evenly redistribute elements in that interval

In fact: intervals grow by walking up complete binary tree on $\Theta(\lg n)$-size chunks of array:



Update:

- update the leaf node ($\Theta(\lg n)$ chunk) containing elt.
- walk up tree until reach node within threshold
- $\text{density(node)} = \dfrac{\#\text{ elements in interval below node}}{\#\text{ array slots in interval}}$
- density thresholds depend on depth d of node:
  - $\text{density} \geq \frac{1}{2} - \frac{1}{4}\frac{d}{h} \in [\frac{1}{4}, \frac{1}{2}]$ ~ not too sparse
  - $\text{density} \leq \frac{3}{4} + \frac{1}{4}\frac{d}{h} \in [\frac{3}{4}, 1]$ ~ not too dense
- evenly rebalance descendant elements in node's interval

Analysis:

- thresholds get tighter as we go up

$\Rightarrow$ rebalancing a node puts children far within threshold: $|\text{density} - \text{threshold}| = \frac{1}{4h} = \Theta(\frac{1}{\lg N})$.

$\Rightarrow$ before this node is rebalanced again, must have $\underbrace{\Omega(\frac{\text{capacity}}{\lg N})}$ updates to bring child out of threshold

$= \Omega(1)$ by our use of leaves of size $\Theta(\lg N)$

$\Rightarrow$ amortized rebuilding caused by update below node $= \Theta(\lg N)$

- each leaf is below $h = \Theta(\lg N)$ ancestors

$\Rightarrow$ amortized rebuilding/update $= \Theta(\lg^2 N)$

Worst-case bounds possible [Willard – I&C 1992; Bender, Cole, Demaine, Farach-Colton, Zito – ESA 2002]

List labeling: related problem

maintain explicit tag for each element in a linked list
such that tags are monotone through list
subject to linked-list updates: insert/delete here

Best results:

tag space	time/update	
$(1+\varepsilon)n \cdots n \lg n$	$O(\lg^2 n)$	= ordered file maintenance
$n^{1+\varepsilon} \cdots n^{O(1)}$	$\Theta(\lg n)$	→ O via modified thresholds; Ω [Dietz, Seiferas, Zhang – SIDMA 2005]
2^n	$\Theta(1)$	→ trivial

Order queries in list: easier problem from L7 (full persistence)

maintain linked list subject to insert/delete here
& query: is node x before node y?

- $O(1)$ solution via indirection: [Dietz & Sleator – STOC 1987; Bender, Cole, Demaine, Farach-Colton, Zito – ESA 2002]

$\boxed{\Theta(n/\lg n)}$ } $\Theta(\lg n)$ solution using tag space $n^{\Theta(1)}$

$\boxed{\Theta(\lg n)}\boxed{\Theta(\lg n)} \cdots \boxed{\Theta(\lg n)}$ } trivial $O(1)$ solution using exponential tag space

- implicit tag of elt. = (top tag, bottom tag) ~ $O(\lg n)$ bits

⇒ can still compare two tags in $O(1)$ time

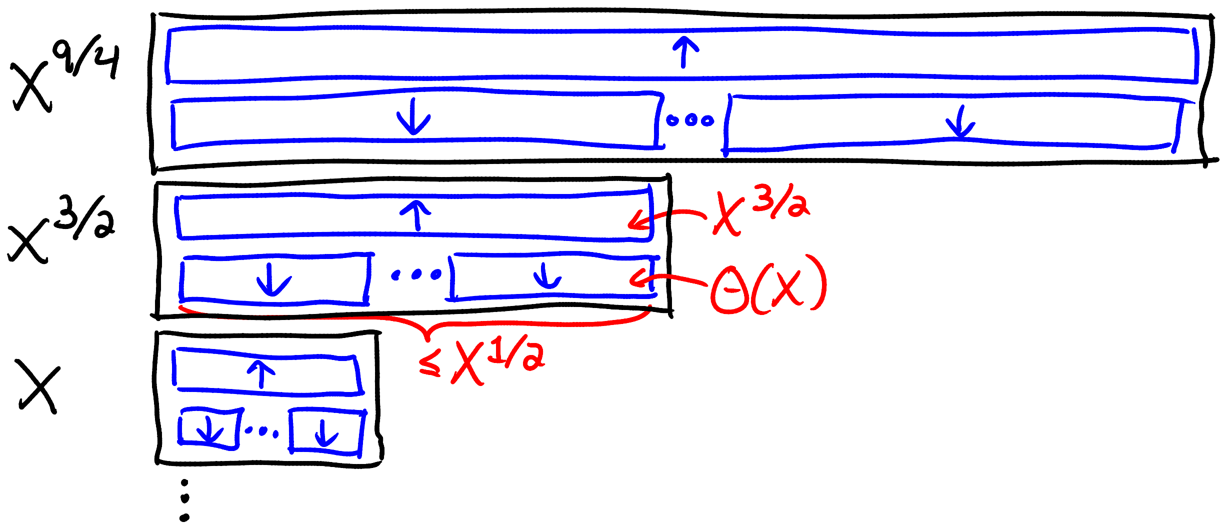
- top updates change many implicit tags at once
- bottom chunks slow updates to top by $\Theta(\lg n)$ factor

⇒ $O(1)$ amortized cost

- worst-case bounds also possible [same refs.]

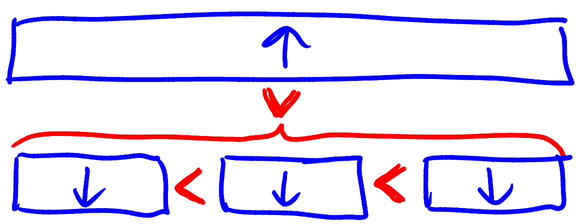
Cache-oblivious priority queue [Arge, Bender, Demaine, Holland-Minkley, Munro - STOC2002/SICOMP; Brodal & Fagerberg - ICALP 2002]

- $\lg \lg n$ levels of size $N, N^{2/3}, N^{4/9}, \ldots, c = O(1)$
- level $X^{3/2}$ has 1 up buffer of size $X^{3/2}$
  & $\le X^{1/2}$ down buffers each of size $\Theta(X)$,
  all except first with $\Theta(X)$ elements



- levels stored in order, say smallest to largest

Invariants:



- keys in down buffers < up buffer at same level
- keys in down buffers @ $X^{3/2}$ < keys in down bufs. @ $X^{9/4}$
- order between down buffers in a level: first < second < ...

Insert:

1. append to smallest up buffer
2. swap into smallest down buffers if necessary
3. if up buffer overflows: push

Push X elements into level $X^{3/2}$

where elts. > all elts. in down bufs. at level X below

1. sort elts.
2. distribute among down buffers (& possibly up buffer):
   - scan elts., visiting down buffers in order
   - when down buffer overflows: split in half, link together
   - when # down bufs. overflows: move last to up buffer
   - when up buffer overflows: push it up to $X^{9/4}$

Delete-min:

1. if smallest down buffer underflows: pull
2. extract smallest elt. in smallest down buffer

Pull X smallest elts. from level $X^{3/2}$ (and above)

1. sort first two down buffers & extract leading elts.
2. if <X: pull $X^{3/2}$ smallest elts. from level $X^{9/4}$ above
   sort these elts. & up buffer
   put larger elts. in up buffer (same # as before)
   extract smallest elts. to get X total smallests
   distribute rest into down buffers

Analysis:

Claim: push/pull at level $X^{3/2}$ (sans recursion) costs $O(\frac{X}{B} \log_{M/B} \frac{X}{B})$ memory transfers

- assume all levels of size $\le M$ stay in cache $\Rightarrow$ free
- tall-cache assumption: $M \ge B^2$ (say)
- push at level $X^{3/2} \ge B^2 \Rightarrow X > B^{4/3} \Rightarrow \frac{X}{B} > 1$
  - sort costs $O(\frac{X}{B} \log_{M/B} \frac{X}{B})$ memory transfers
  - distribute costs $O(\underbrace{\frac{X}{B}}_{\text{scan}} + \underbrace{X^{1/2}}_{\text{startup each down buffer}})$ memory transfers
  - if $X \ge B^2$ then cost $= O(\frac{X}{B})$
  - else: only one such level with $B^{4/3} \le X \le B^2$
    can keep 1 block per down buffer in cache:
    $X \le B^2 \Rightarrow X^{1/2} \le B \le \frac{M}{B}$ by tall cache
    so just pay $O(\frac{X}{B})$ at this level too
- pull at level $X^{3/2} \ge B^2$:
  - sort costs $O(\frac{X}{B} \log_{M/B} \frac{X}{B})$ memory transfers
  - another sort of $X^{3/2}$ elts. only when recursing
    $\Rightarrow$ charge to recursive pull

Totaling: X elts. involved in push/pull costing $O(\frac{X}{B} \log_{M/B} \frac{X}{B})$
- each elt. goes up & then down (more or less)
  - real proof messier

$\Rightarrow O(\frac{1}{B} \sum_X \log_{M/B} \frac{X}{B})$ amortized cost per element

exp. geometric $\quad$ geometric

$= O(\frac{1}{B} \log_{M/B} \frac{N}{B})$