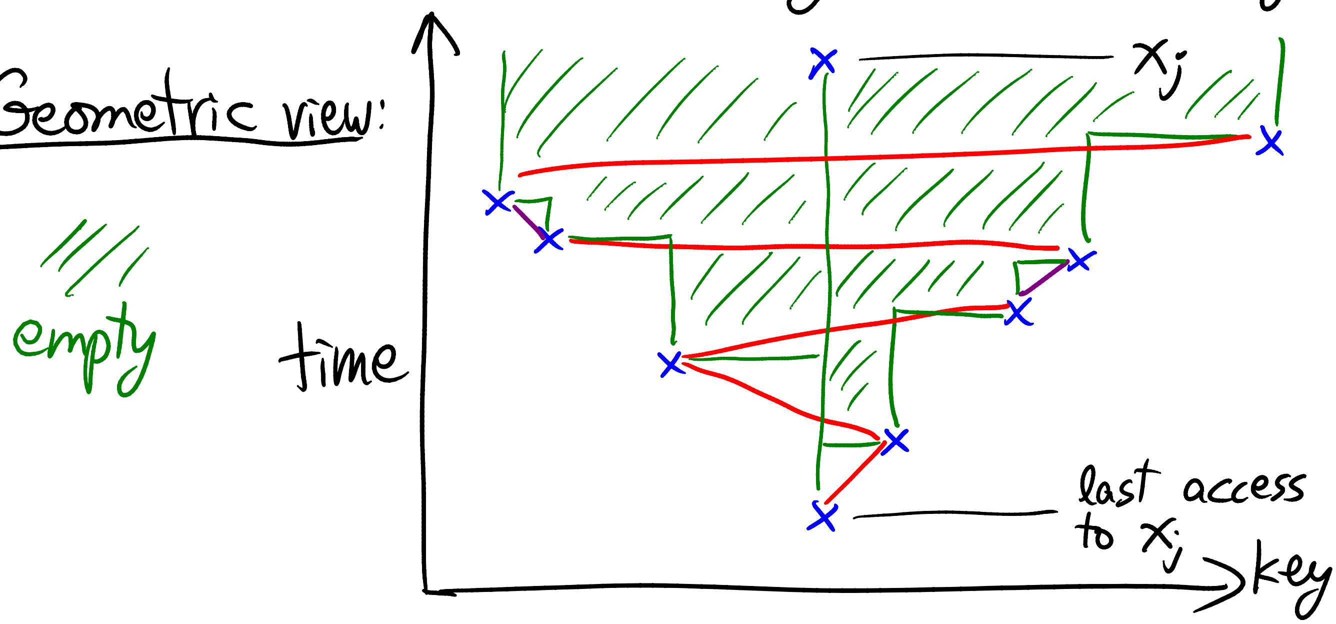
Lecture 3

Wilber's second lower bound [Wilber–SICOMP 1989]

wilber2(x_j):

- look at where x_j fits among $x_i, x_{i+1}, \ldots, x_{j-1}$ for $i = j-1, j-2, \ldots$ until previous access to x_j
- say $a_i < x_j < b_i$ is the tightest pair for i
- as i decreases, consider more x's
  $\Rightarrow$ bounds tighten: a_i increases & b_i decreases
- wilber2(x_j) = # alternations between a_i increasing & b_i decreasing

Geometric view:



- $\text{wilber2}(x) = \sum_{j=1}^{m} \text{wilber2}(x_j)$

  is a lower bound on all BSTs for x
  [proof somewhat similar to wilber1 proof below]

<u>Key-independent optimality</u>: [Iacono – ISAAC 2002]

- suppose key values are "meaningless"

$\Rightarrow$ might as well permute them uniformly at random

- then $E_\pi[\text{dynamic OPT}(\pi(x_1), \pi(x_2), \dots, \pi(x_n))]$

$$= \text{working-set bound } \Theta\left(\sum_{i=1}^{m} \lg t_i(x_i)\right)$$

$\Rightarrow$ splay trees, etc. are key-independently optimal

<u>Proof sketch</u>: O trivial; Ω via wilber2

- wilber2(x_j) just considers working set of x_j:

  $W = \{$elements accessed since last access to $x_j\}$

  - only one access to x_i, $i<j$, plays a role
- π permutes this working set
- x_j falls roughly in the middle of W
- $E[\#\text{ times } a_i \text{ increases}] = \Theta(\lg |W|)$
  - like $E[\#\text{ times max. changes in prefixes of random list}] = \sum_{i=1}^{k} 1/i$
- $E[\#\text{ times } b_i \text{ increases}] = \Theta(\lg |W|)$
- <u>claim</u>: expected constant fraction of these changes interleave

$\Rightarrow \text{wilber2}(x_j) = \Theta(\lg |W|) = \Theta(\lg t_j(x_j))$ $\square$

$\boxed{\text{OPEN: wilber2} = \Theta(\text{dynamic OPT})?}$

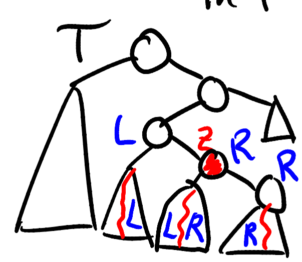
Wilber's first lower bound: [Wilber–SICOMP 1989]

- fix a lower-bound tree on same keys
  e.g. perfect binary tree
- for each node y of P:
  - label each access x_i as
    L if key x_i is in y's left subtree
    R if key x_i is in y's right subtree
    blank if $x_i = y$ or x_i outside y's subtree
  - interleave(y) = # alternations between L & R
- wilber1(x) = interleave(x) = $\sum_y$ interleave(y)

is a lower bound on all BSTs for x

Proof sketch:

- define transition point of node y in P to be highest node z in BST T such that root-to-z path includes nodes from left & right subtrees of y in P
  - well-defined
  - doesn't change until z touched
  - different for different y



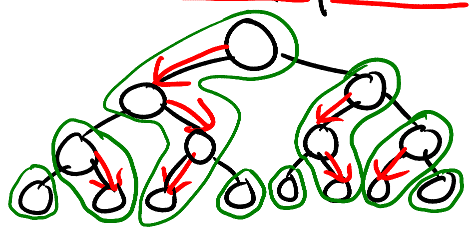
- must touch transition point of y in next L or R access i.e. within next 2 interleaves

$\Rightarrow$ must touch $\geq$ interleave(x)/2 nodes. □

$O(\lg \lg n)$-competitive BST: Tango trees

[Demaine, Harmon, Iacono, Pătraşcu — FOCS 2004, SICOMP 2007]

- define preferred child of node y in P to be
  - left if accessed left subtree(y) more recently
  - right if accessed right subtree(y) more recently
  - none if no access to either subtree yet
- preferred path = chain of preferred child pointers
  - partition of nodes in P

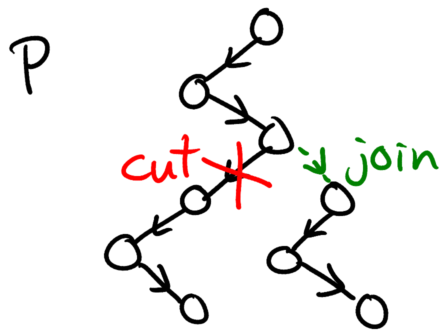


- idea: store each preferred path in auxiliary tree:
  - conceptually separate balanced BST
  - leaves link to roots of aux. trees of children paths
- search starts at top aux. tree (containing root(P))
  - aux. tree has $\le \lg n$ nodes [if height(P) $= \lg n$]
  - $\Rightarrow$ search in $O(\lg \lg n)$ time
  - each jump to next aux. tree = nonpreferred edge = preferred edge change = interleave
  - visit k aux. trees $\Rightarrow$ interleave $\ge k-1$
  - $O(k \lg \lg n)$ search cost, $\Omega(k)$ lower bound
  - $\Rightarrow O(\lg \lg n)$-competitive (modulo update cost)

- dynamic finger aux. trees $\Rightarrow$ get $O(k \lg \frac{\lg n}{k})$
- $\Rightarrow$ cost $= O\left(\text{OPT} \cdot \lg \frac{\lg n}{\text{OPT}/n}\right)$ ... actually wilber1, not OPT
  - tightest possible bound in terms of n & wilber1 (by entropy)

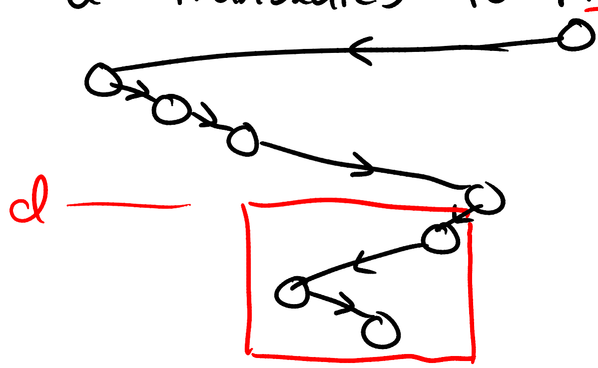
Auxiliary trees:

- changing a preferred child corresponds to cutting one path & joining two paths:



- cut path into nodes with P depth $\le d$ & $> d$
- join two paths with consecutive depths
- like split & concatenate in balanced BSTs
  (immediate pointer-machine DS)
- except aux. trees ordered by key, not depth

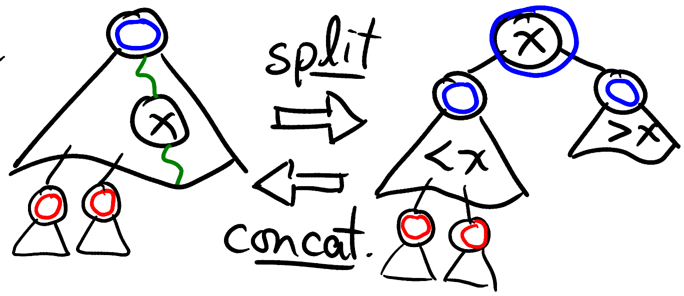
$\Rightarrow$ depth $> d$ translates to interval of keys



- extraction/insertion of interval of keys can be implemented by $O(1)$ splits & concatenates
  $\Rightarrow O(\lg |\text{aux. tree}|) = O(\lg \lg n)$ time/interleave

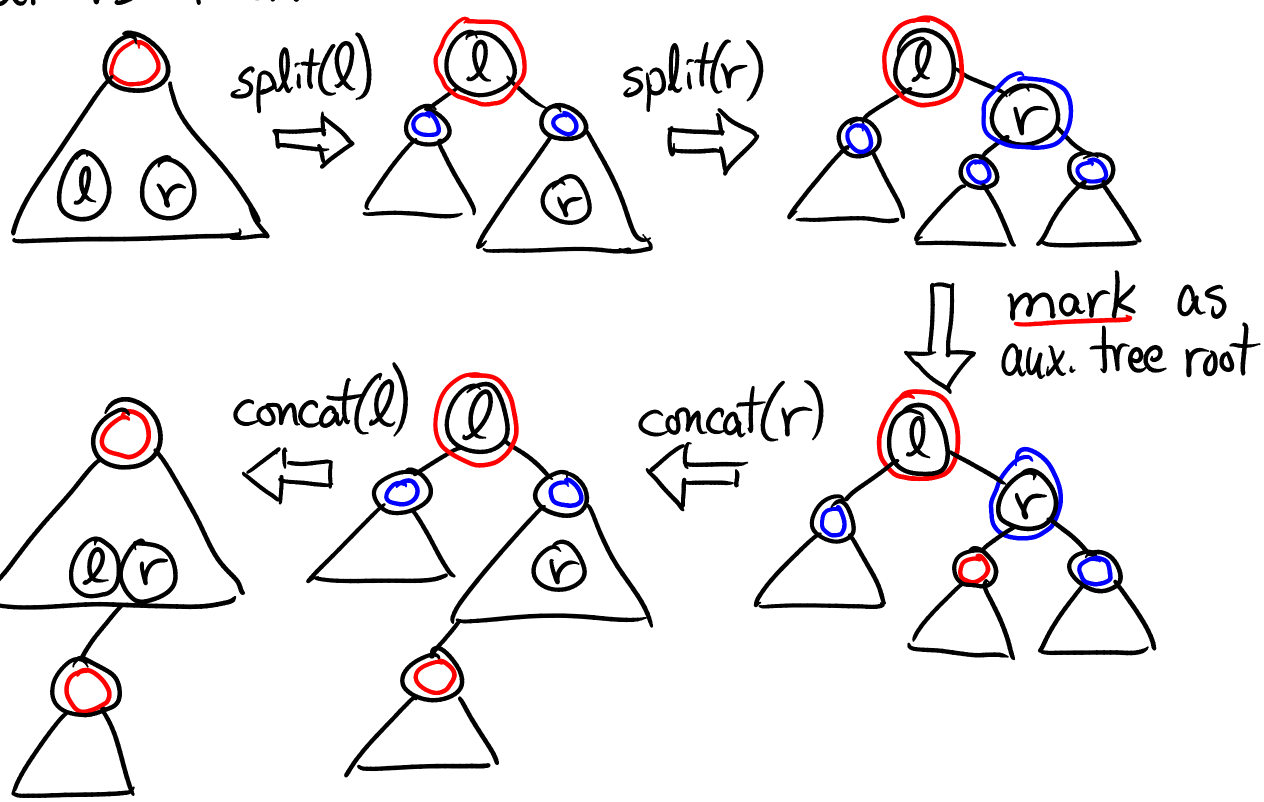
Combining aux. trees into one BST:

- each aux. tree is a contiguous subtree
- mark root of each aux. tree
- define



modifications of standard logarithmic split & concat to use temporary roots, ignoring children trees

- cut is then:



- join is the reverse
- need depth augmentation to find l & r