Binary search trees (BSTs)
- again focus on access(x), keys \{1, 2, ..., n^3\}

Models:
1. **worst case**: \(\Theta(\log n)\)
   - *LB*: access deepest node \(\Rightarrow\) depth \(\geq \log n\)
   - *UB*: red-black trees, AVL trees, ...
2. **stochastic/static**: \(p_i = f_i / m\)
   - dynamic program for OPT: \([\text{Gilbert & Moore 1959}]\)
     - guess which key \(r\) to put at root (\(n\) choices)
     \(\Rightarrow\) two subproblems: left \((< r)\) & right \((> r)\)
     - in general, subproblem = interval of keys
     \(\Rightarrow\) \(O(n^2)\) subproblems
     - cost(problem) = 1 + \(Pr_{\text{left}}\cdot \text{cost(left)}\)
       + \(Pr_{\text{right}}\cdot \text{cost(right)}\)
   \(\Rightarrow\) \(O(n^2)\) time
   - smarter DP runs in \(O(n^3)\) time \([\text{Knuth 1971}]\)

**OPEN**: \(o(n^3)\) time possible?

- static OPT between \(H - \log H\) & \(H + 3\)
  where entropy \(H = \sum_i p_i \log 1 / p_i\)
  \([\text{Huffman trees achieve } H + O(1) \text{ but not BST}]\)
(3) dynamic: one-finger model
   - finger starts at BST root for each access
   - pay 1 for each left, right, parent move
     or rotate [with parent]
   - must touch x

BST access algorithms:

1. Transpose analog: rotate x once toward root
   - for uniform random accesses,
     all BSTs equally likely in limit
   ⇒ expected height \( O(\sqrt{n}) \)
     \[\text{[Allen & Munro - JACM 1978]}\]

2. Move To Root: rotate x until it's at root
   \( = 2 \ln 2 \approx 1.38 \cdot \text{stochastic OPT} \)
   \( \neq O(1) \cdot \text{static OPT} \)
3. Splay trees: variation on MTR [Sleator & Tarjan-1985]

- Zig-zig case:
  \[
  \begin{array}{c}
  Z \quad \xrightarrow{\text{rotate } y} \quad Z \\
  y - x - y \\
  \end{array}
  \]

- Zig-zag case:
  \[
  \begin{array}{c}
  y - x - y \\
  \end{array}
  \]

- Key feature: ≤ half nodes on x path go down
  - At end, may need single rotation
  - \( O(1) \) static OPT

\[
\text{OPEN: } = O(1) \cdot \text{dynamic OPT?}
\]

\[ \text{dynamic optimality conjecture} \]
Access lemma: just count rotations
- assign arbitrary weight \( w_i \) to each elt \( i \)
- define \( W(x) = \text{sum of weights of keys in subtree rooted at } x \)
- potential \( \Phi = \sum_x \log W(x) \)

Eg. \( w_i = 1 \) \( \forall i \) balanced \( \Rightarrow \Phi = \Theta(n) \)
path \( \Rightarrow \Phi = \Theta(n \log n) \)

- amortized cost of one splay step \((\text{zig-zag})\)
  \( \leq 3 \cdot \left[ \log W^{\text{new}}(x) - \log W^{\text{old}}(x) \right] \)
  (some checking + concavity of \( \log \))
  (note rotation changes only \( W(x) \) & \( W(x)'s \ parent \))

- amortized cost of \( \text{access}(x) \) telescopes:
  \( \leq 3 \cdot \left[ \log W^{\text{new}}(x) - \log W^{\text{old}}(x) \right] + 1 \) \( = W(\text{tree}) \) final single rotation

- \( \min \Phi \geq \sum_i \log w_i \); \( \max \Phi \leq n \log W(\text{tree}) \)

- \( \max \Phi - \min \Phi \leq \sum_i \left[ \log W(\text{tree}) - \log w_i \right] \)
  bound on am. cost to \( \text{access}(i) \)

- assuming \( m = \Omega(\text{this startup cost}) \),
  \( \text{access}(x) \) costs \( O(\log W(\text{tree}) - \log W^{\text{old}}(x)) \)
  \( = O(\log W(\text{tree}) - \log W_x) \)
**Consequences of access lemma:**

1. **Logarithmic:** \( w_i = 1 \ \forall i \Rightarrow W(\text{tree}) = n \)
   \( \Rightarrow O(\lg n) \)
   
   **Startup cost:** \( O(n \lg n) \)

2. **Static Optimality:** \( w_i = p_i \Rightarrow W(\text{tree}) = 1 \)
   \( \Rightarrow O(1 - \lg p_x) = O(\lg \frac{1}{p_x}) \)
   
   **Startup cost:** \( O(H) \)

3. **Static Finger Theorem:** Fix "finger" at a node \( f \)
   \( \frac{1}{1 + (i - f)^2} \Rightarrow W(\text{tree}) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \)
   \( \Rightarrow O(1 + \frac{1}{(i - f)^2}) = O(\lg [\frac{a + |i - f|]}{\text{distance to finger}}) \)
   
   **Startup cost:** \( O(n \lg n) \)

4. **Working-set Theorem:** \( O(1 + \lg t_i(x_i)) \)
   
   Where \( t_i(x) = \# \text{distinct } y \text{s accessed since last access to } x \text{ before time } i \) (including \( y \) itself)
   
   \( w_x \) at time \( i = \frac{1}{t_i(x)^2} \Rightarrow W(\text{tree}) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \)
   \( \Rightarrow O(1 + \lg t_i(x_i)^2) = O(1 + \lg t_i(x_i)) \) for access
   
   \( \text{next } t_{i+1}(x_i) = 1 \Rightarrow w_{x_i} \uparrow 1 \)
   
   Some \( t_{i+1}(y) = 1 + t_i(y) \Rightarrow w_y \downarrow \text{slightly} \)
   
   \( \Rightarrow \) no \( w_y \) increases except \( w_{x_i} \)
   
   \( \Rightarrow \) no \( w(y) \) increases, even \( W(x_i) = W(\text{tree}) \) (fixed)
   
   \( \Rightarrow \Phi \) does not increase from reweighting
   
   **Startup cost:** \( O(n \lg n) \)
More bounds:
5. **Scanning theorem**: (a.k.a. sequential access theorem)
   access sequence 1, 2, ..., n costs \( O(n) \)
   from any initial tree  
   [Tarjan - Combinatorica 1985]
6. **Dynamic finger theorem**:  
   \( \text{access}(x_i) \) costs \( O(\log(2 + |x_i - x_{i-1}|)) \)
   amortized
   (assuming \( m \geq n \))

OPEN:
7. **Deque conjecture**:  
   [Tarjan - Combinatorica 1985]
   two fingers \( f_1 \leq f_2 \), each access either ++\( f_i \) or --\( f_i \)
   \( \Rightarrow O(m+n) \) total cost
   \(- O((m+n) \times (m+n)) \)
   [Sundar - Combinatorica 1992]
8. **Traversal conjecture**:  
   [Sleator & Tarjan - JACM 1985]
   access elts. from preorder traversal of fixed BST
   costs \( O(n) \), from any initial tree
9. **Monotonicity conjecture**:  
   [Iacono - personal comm.]
   removing an access into the sequence
   never increases the total cost
10. **Unified conjecture**:  
    [Iacono - SODA 2001]
    \( \text{access}(x_i) \) costs \( O(\log \min_y [t_i(y) + |x_i - y| + 2]) \)
    amortized
    “fast if close to something recent”
    —achievable by pointer-machine DS  
    [Iacono - SODA 2001; Bädoiu, Cole, Demaine, Iacono - Algorithmica]
Bound implications: [Iacono - SWAT 2000 & SODA 2001]

unified → working set → static optimality → dynamic finger → static finger