6.849: Geometric Folding Algorithms, Spring 2017

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Problem Set 6 Solutions

Due: Wednesday, March 22, 2017

Solve Problem 6.1 and *either* Problem 6.2 or 6.3.

Problem 6.1 [Mandatory, Collaboration OK]. On Problem Sets 1–6, we will ask you to write a problem (solved or unsolved) related to the material covered in class. The problem should be original to the best of your knowledge, so be creative and diverse! Folding can be applied to mathematics, computation, engineering, architecture, biology, and beyond, so write a problem that is related to a field that interests you. If you write a problem whose solution can be solved from the material covered in class, then we may adapt your problem for future problem sets. If you pose a problem whose solution is not yet known, we may try to solve it in class during our open problem sessions, or it may become inspiration for a class project. Feel free to include solutions or commentary for your problem. While writing a problem is required, your submission will be graded generously, so have fun and share with us your exploration of the course material.

(On future problem sets, we will instead be asking about progress on your project.)

Solve ONE of the two problems below.

Problem 6.2 [Collaboration OK]. Analyze how many bars are used in (the corrected) Kempe's linkage construction from Lecture 9. Specifically, given a target polynomial $f \in \mathbb{R}[x, y]$ of total degree n,[1] and a closed disk B in the plane, show that the constructed linkage that traces $S = B \cap \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ uses $O(n^2)$ bars.

Solution by Alex Padron:

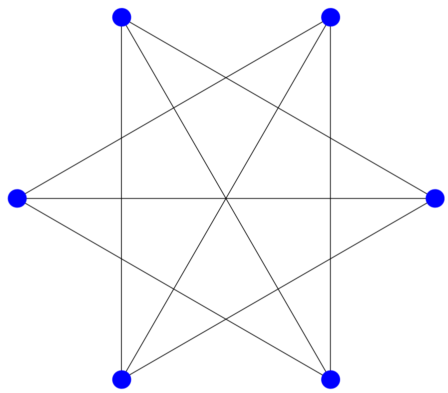
We start with an degree n polynomial. Therefore, after substituting $x = \frac{r}{2}\cos\alpha + \frac{r}{2}\cos\beta$ and $y = \frac{r}{2}\sin\alpha + \frac{r}{2}\sin\beta$, and simplifying to only cosines, the degree of every cosine term will be at most n.

We now see that the total number of unique cosine terms is $O(n^2)$. By unique we mean that the cosine will have some term inside, $A \cdot x + B \cdot y + C \cdot \frac{\pi}{2}$. The $C \cdot \frac{\pi}{2}$ is just to handle converting between sines and cosines. A cosine term is unique if its A, B, and C are not all shared by another cosine term. Moreover, it is clear to see that if a set of cosine terms are not unique, we can simply add them together to get one unique term. Now we want to examine the possible values each of A, B, C can take. We see that A and B can only take values in the range $[-n, n]$. The reason for this is that each time we multiply two unique cosine terms, the range of those two terms adds together, i.e., the new bounds are just the max of the previous bounds. We can see this by looking at the formula $(\cos x) \cdot (\cos y) = \frac{1}{2}(\cos(x + y) + \cos(x - y))$. To generate the new bounds for A and B here, we can get a new upper bound by adding the two original upper bounds in the term $\cos(x + y)$, and a new lower bound by adding the original lower bounds in the same term. Also note that if the lower bound is positive for both, as it is in the starting equation for x and y, then the lower bound will just be the lower bound of one subtracted from the lower bound of the other. This suffices to show that A and B will both exist in the range $[-n, n]$. C is either 0 or 1, since

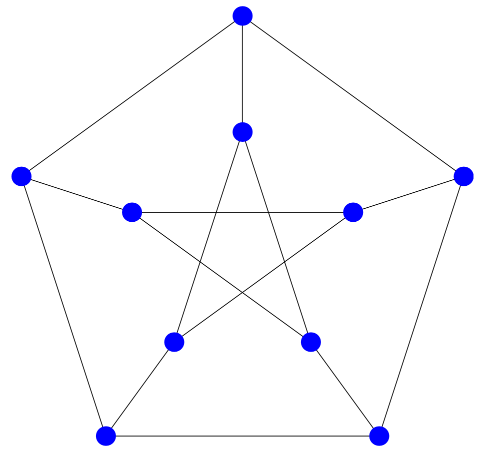
[1] The *total degree* of a polynomial $\sum_i c_i x^{a_i} y^{b_i}$ is $\max_i(a_i + b_i)$.

other values degenerate into one of these two cases. Therefore, the total number of unique cosine terms is the product of the possibilities for A, B, and C, which is $O(n^2)$.

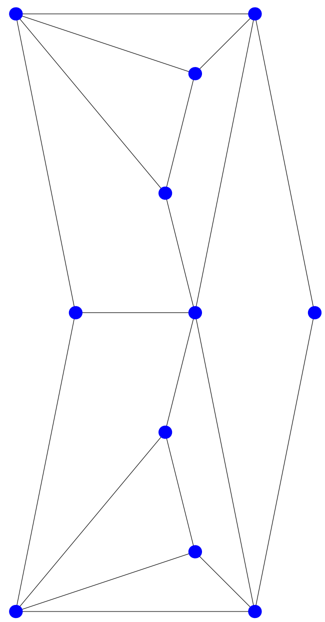
Problem 6.3 [Collaboration OK]. For each of the following graphs, characterize as generically flexible, minimally generically rigid, or redundantly generically rigid.



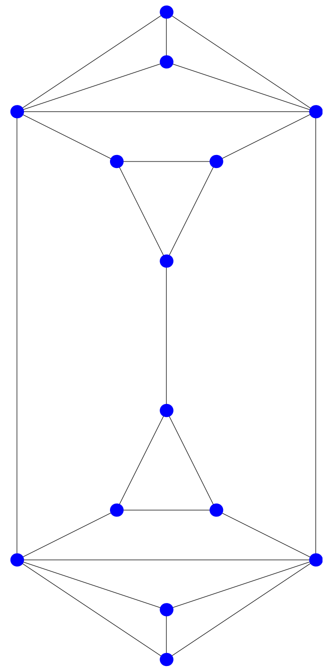
(a)



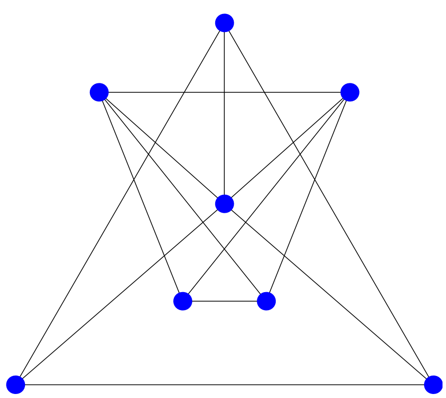
(b)



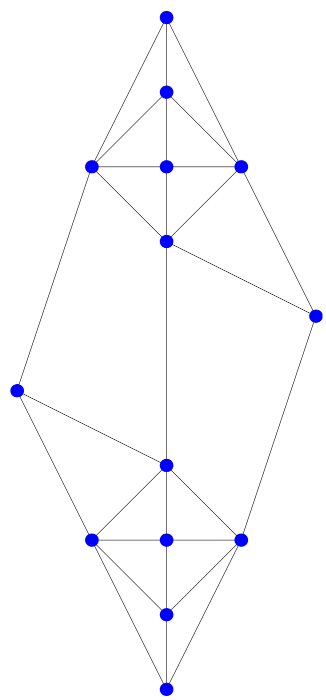
(c)



(d)



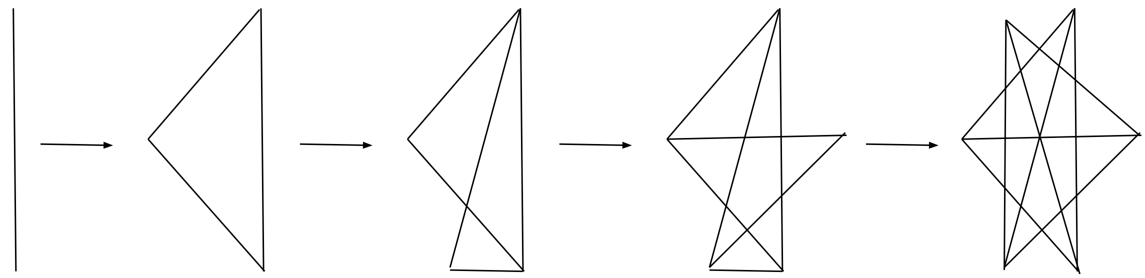
(e)



(f)

Solution by Anna Ellison:

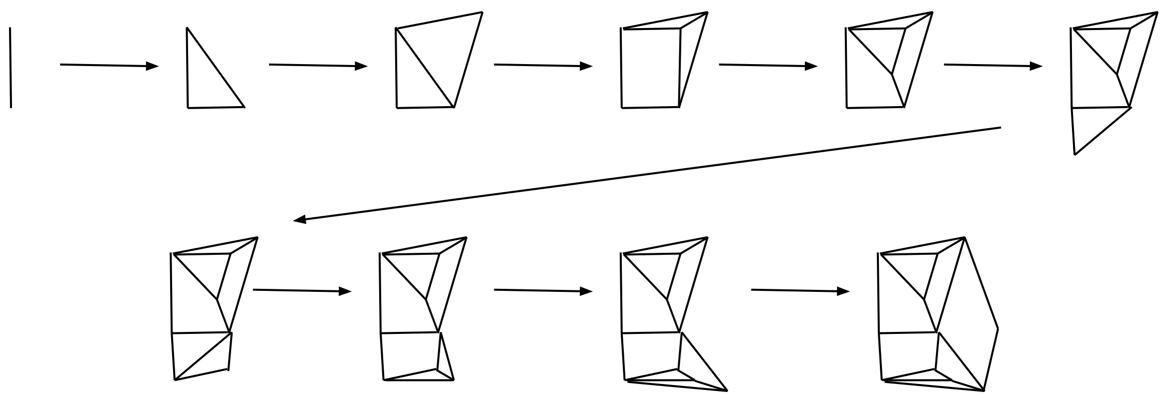
(a) This is minimally generically rigid.



This is a Henneberg construction that shows it is minimally generically rigid.

(b) This has 10 vertices and $15 < 2 \cdot 10 - 3$ edges, so it will be generically flexible.

(c) This is minimally generically rigid. It can be constructed as follows:



(d) This is generically flexible. If it were generically rigid, then the graph would also be rigid if we removed one edge from the top four vertices, since the top 4 vertices would still be rigid. However, the new graph has 24 edges and 14 vertices, and $24 < 2 \cdot 14 - 3 = 25$, so it must be generically flexible. Then, the first graph must have been generically flexible.

(e) This is generically flexible: if you remove the center vertex, there are two connected components, so one of those components can rotate around the center vertex relative to the other component.

(f) This is redundantly generically rigid. The following construction shows that a linkage with a subset of edges is a minimally generically rigid graph.