6.849: Geometric Folding Algorithms, Spring 2017

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Problem Set 2 Solutions

Due: Wednesday, February 22, 2017

Solve Problem 2.1 and *either* Problem 2.2 or 2.3.

Problem 2.1 [Mandatory, Collaboration OK]. On each problem set, we will ask you to write a problem (solved or unsolved) related to the material covered in class. The problem should be original to the best of your knowledge, so be creative and diverse! Folding can be applied to mathematics, computation, engineering, architecture, biology, and beyond, so write a problem that is related to a field that interests you. If you write a problem whose solution can be solved from the material covered in class, then we may adapt your problem for future problem sets. If you pose a problem whose solution is not yet known, we may try to solve it in class during our open problem sessions, or it may become inspiration for a class project. Feel free to include solutions or commentary for your problem. While writing a problem is required, your submission will be graded generously, so have fun and share with us your exploration of the course material.

Solve ONE of the two problems below.

Problem 2.2 [Collaboration OK]. Which of the four crease patterns on the following page are flat foldable? Are any simply foldable (foldable by a sequence of simple folds)? Justify each answer by either submitting a flat folding or arguing why the crease pattern cannot fold flat.

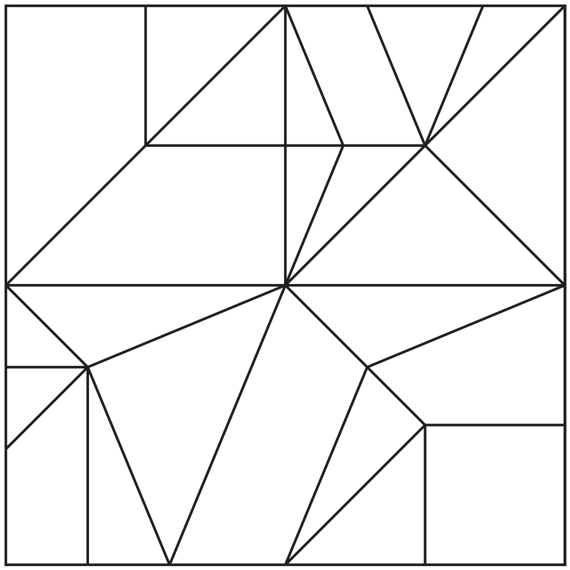
Solution adapted from Laphonchai Jirachuphun

The first three are not flat-foldable. The last is flat foldable via simple folds.

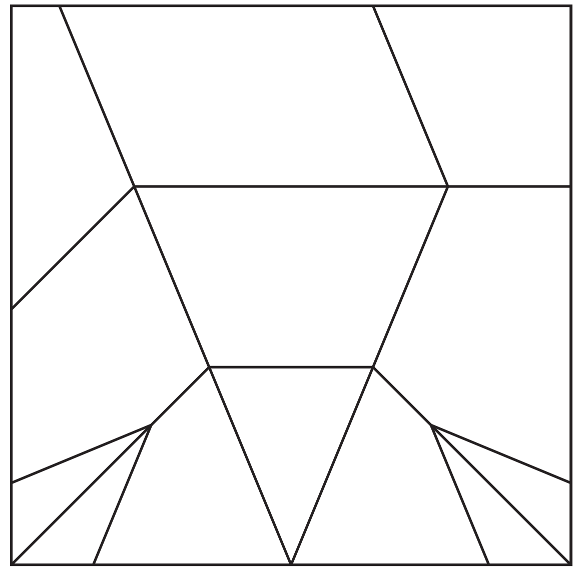
(1) Not flat-foldable. Consider the middle vertex on the crease pattern, pictured in Figure 1a. Using Kawasaki's Theorem (1989), the sum of the odd angles must equal that of the even angles for the paper to be flat foldable. However, we can clearly see from the figure that the sum of the even angles (✓) is larger than the sum of the odd angles ($\times$). Hence, it is not flat foldable.

(2) Not flat-foldable. First, consider the top middle vertex A; refer to Figure 2(a). Since the left and right angle are larger than $\angle BAC$, AB and AC have to be given different mountain-valley assignments. We will only consider the top part of the paper, so we can assume that the paper is symmetric about the middle vertical axis. Hence, without loss of generality, let AB be the valley crease and AC be the mountain crease resulting in Figure 2(b). Now consider the horizontal line $DFBCGE$: it has to be an all-layers fold with either mountain or valley fold. Since the paper is symmetric, we will assign a mountain fold resulting in Figure 2(c). Now, we can see that only one of the vertex G and F is flat foldable, depending on what assignment you give to the line $DFBCGE$. In the case shown in Figure 2(c), F is foldable but not G because there is a layer of mountain fold from $DFBCGE$ right behind it; see Figure 2(d). Therefore, it is not flat foldable.

(Editor comment: the fold transforming Fig 2(c) to (d) is not a simple fold because the layers being folded are not top-most or bottom-most layers, as is required for simple folding. The primary reason that (2) is not foldable by a sequence of simple folds is that no non-intersecting flat folded state exists. Because AB and AC must have different assignment as argued above, one of BD or CE must be on the outside of BC. If BD is on the outside, all creases incident to F cannot fold



(1) http://courses.csail.mit.edu/6.849/spring17/psets/ps2-cp1.pdf



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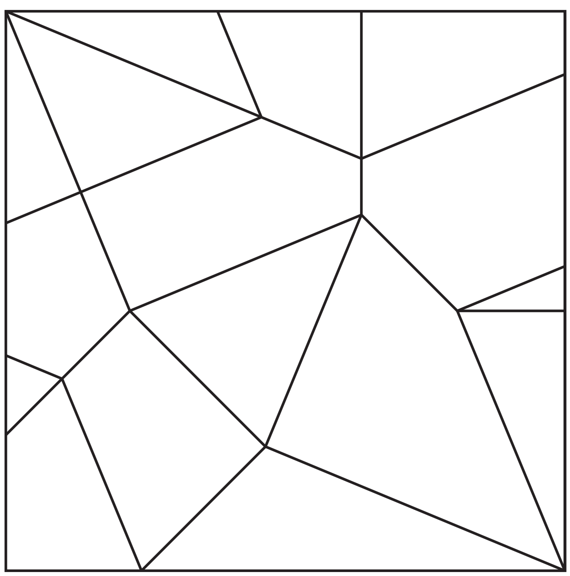
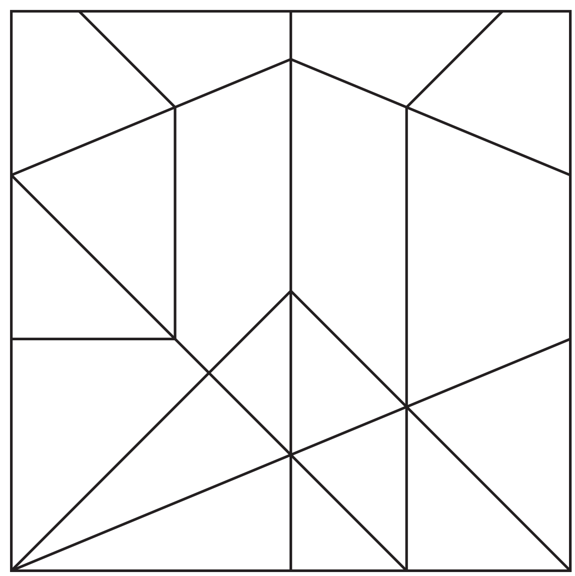


(3) http://courses.csail.mit.edu/6.849/spring17/psets/ps2-cp3.pdf



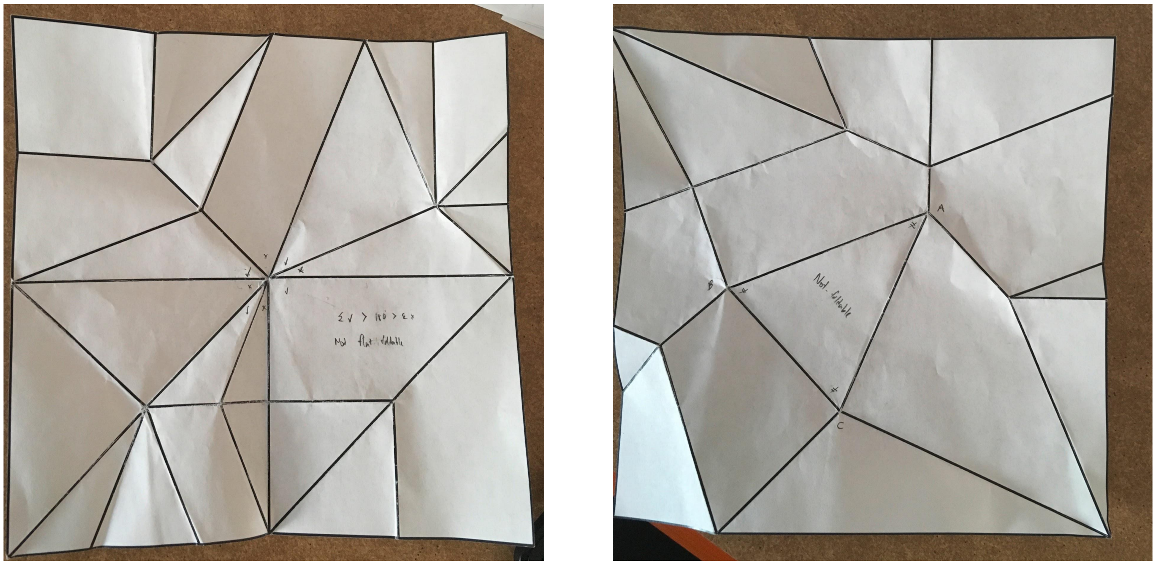
(4) http://courses.csail.mit.edu/6.849/spring17/psets/ps2-cp4.pdf

without also creating new folds on triangle ABC, and similarly for CE. Thus, regardless of what folding operations are allowed, no flat-folded state exists, so (c) is not (globally) flat foldable.

(3) Not flat-foldable. Consider the triangle ABC in Figure 1b. Since all the angles in the triangle are all smallest locally within their single-vertex, by the definition of local foldability from Bern and Hayes (1996), all the sides of the triangle have to have different assignment (either mountain or valley), which is not possible. Hence, it is not flat foldable.

(4) Flat foldable by simple folds.

(Editors comment: An unassigned crease pattern foldable by a sequence of n simple folds could have many valid folded states. So, instead of providing some specific folded state of the crease pattern, we will number creases according to which simple fold they appear in some valid sequence



(a) Diagram for Problem 2.2, Part (1).

(b) Diagram for Problem 2.2, Part (3).

Figure 1: Unfoldability of crease patterns in Problem 2.2.

of simple folds. Figure 3 shows such a labeled seqeuence and can be easily verified.

Problem 2.3 [Collaboration OK]. In class, we saw NP-hardness of the 1D "ruler folding" problem. Now consider the related problem about single-vertex flat foldability. Given an (unassigned) single-vertex *hinge* pattern (a crease pattern where all creases are optional), decide whether there is any flat folding that folds at least one of the hinges. Prove that this problem is weakly NP-hard.

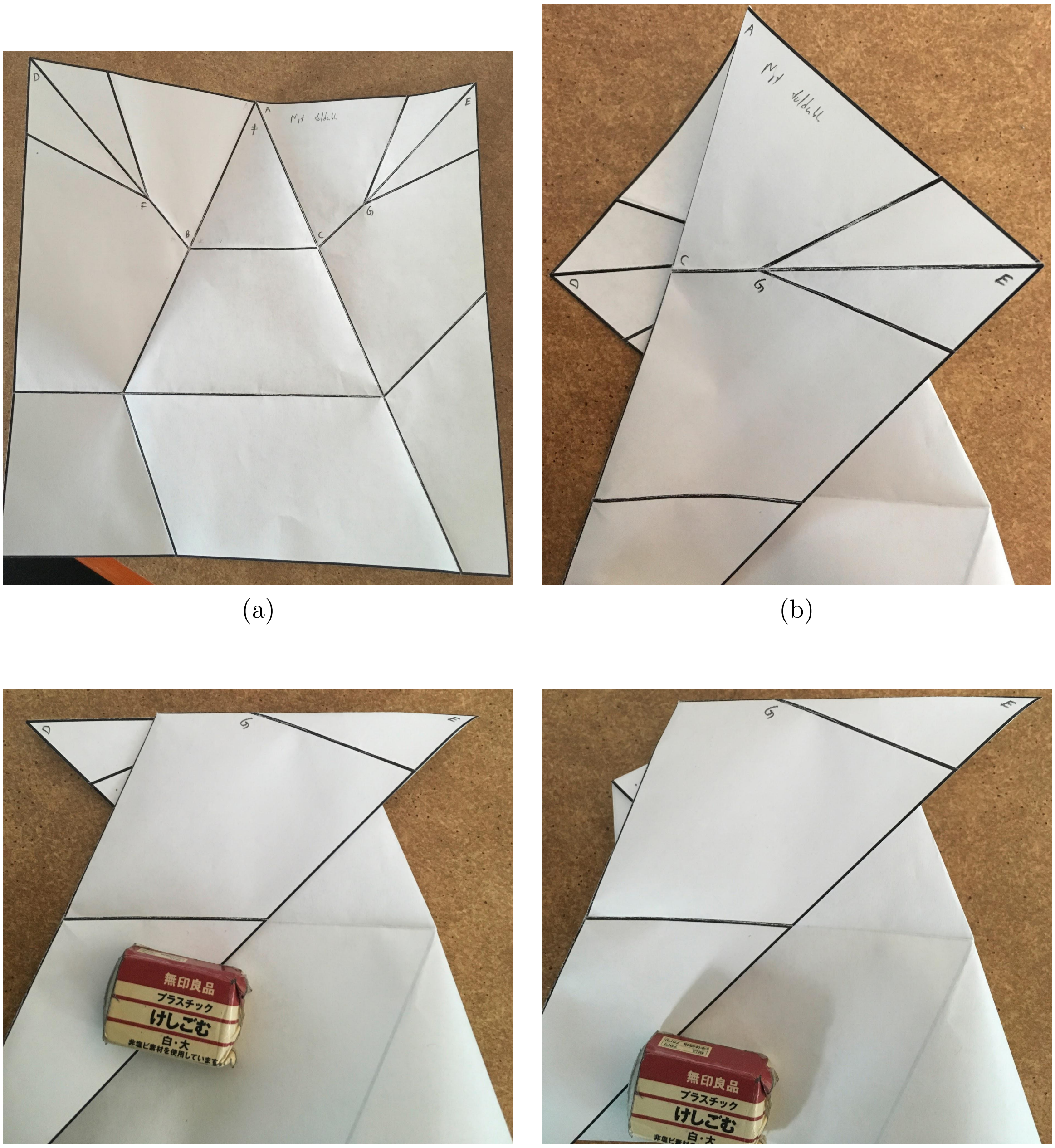
Solution by William Kretschmer

We show (weak) NP-hardness of single-vertex hinge pattern flat-foldability by reduction from PARTITION, the problem of partitioning a multiset of integers into two multisets that sum to the same total. Given a PARTITION instance with integers $a_1, \ldots, a_n$ and $\sum_{i=1}^n a_i = K$, we produce the following hinge pattern:

The angles in Figure 4 are listed up to proportionality; they should all be scaled by a factor of $\frac{2\pi}{3K}$ radians (so e.g. segments marked with K have angle $\frac{2\pi}{3}$ radians). We let O denote the hinge between a_0 and a_{n+1}.

We claim that $a_1, \ldots, a_n$ have a valid partition if and only if this hinge pattern has a nontrivial flat-folding. In one direction, suppose there is a valid partition of $a_1, \ldots, a_n$ into sets S_1 and S_2 both of sum $\frac{K}{2}$. Formally, we also assign a_0 to S_1 and a_{n+1} to S_2. Then, we explicitly leave unfolded all hinges between adjacent segments that are assigned to the same set in the partition. If we designate the remaining hinges as a crease pattern, then they satisfy Kawasaki's theorem for flat foldability. In particular, there must be an even number of hinges, because each hinge marks a transition between S_1 and S_2 (and at least one segment belongs to each of S_1 and S_2). Additionally, the alternating sum and difference of (combined) segment angles equals zero, because it totals to $\frac{2\pi}{3K}(K + \sum S_1 - \sum S_2 - K)$.

In the other direction, suppose this pattern is nontrivially flat foldable. Observe that the O must be folded, because otherwise we would have a single segment with angle $\frac{4\pi}{3}$, a violation of Kawasaki's theorem. Thus, the hinge between a_0 and a_1 must perfectly align with the hinge



(a) (b)

(c) (d)

Figure 2: Diagram for Problem 2, Part (2).

between a_n and a_{n+1}. Furthermore, the crease along O must be an extreme crease, because the sum of a_1 through a_n cannot overlap with more than $\frac{2\pi}{3}$ radians, which is not enough to reach O. We then follow the path along the edge of the disk between the two overlapping hinges. For any segment wherein we move in the counterclockwise direction, we assign the corresponding a_i to S_1 (and conversely, we add clockwise segments to S_2). The total clockwise and counterclockwise displacements relative to the overlapping hinges must be equal because the pattern is flat-folded. For example, in the picture below, we have $\sum S_1 = a_1 + a_4 = a_2 + a_3 + a_5 = \sum S_2$ as a valid partition. In other words, we partition the a_i into two sets S_1 and S_2 such that $\sum S_1 = \sum S_2$. This

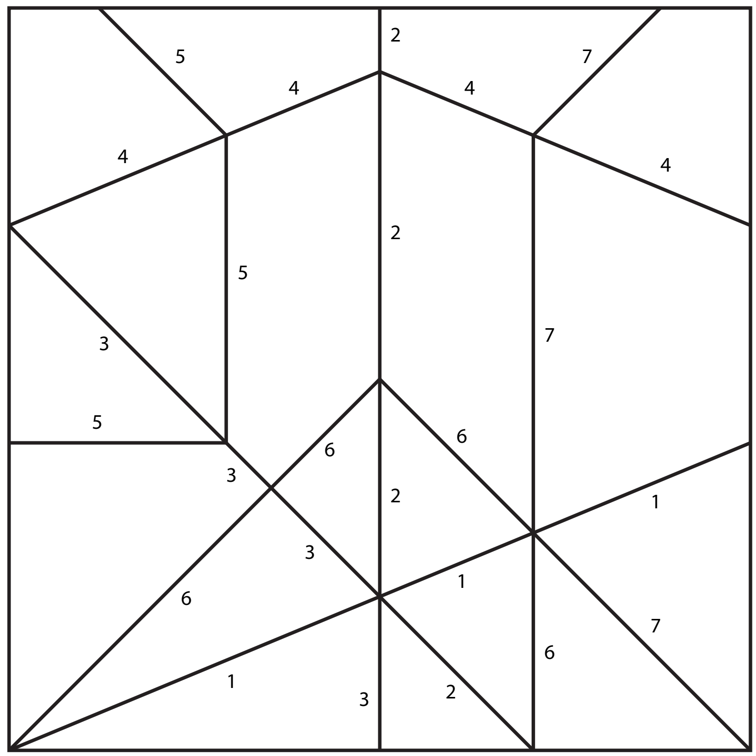


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Figure 3: A sequence of simple folds that can fold example 1(4).

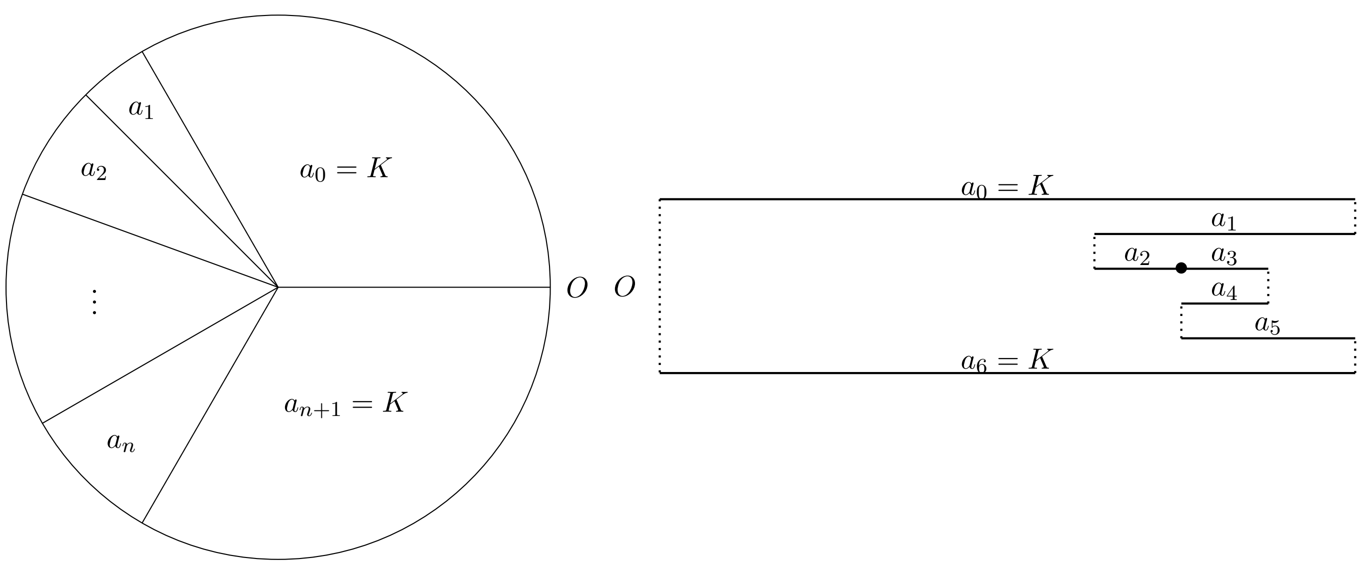

Figure 4: A diagram of the crease pattern (left) and represented alignment problem (right).

shows that the corresponding PARTITION instance is satisfiable.