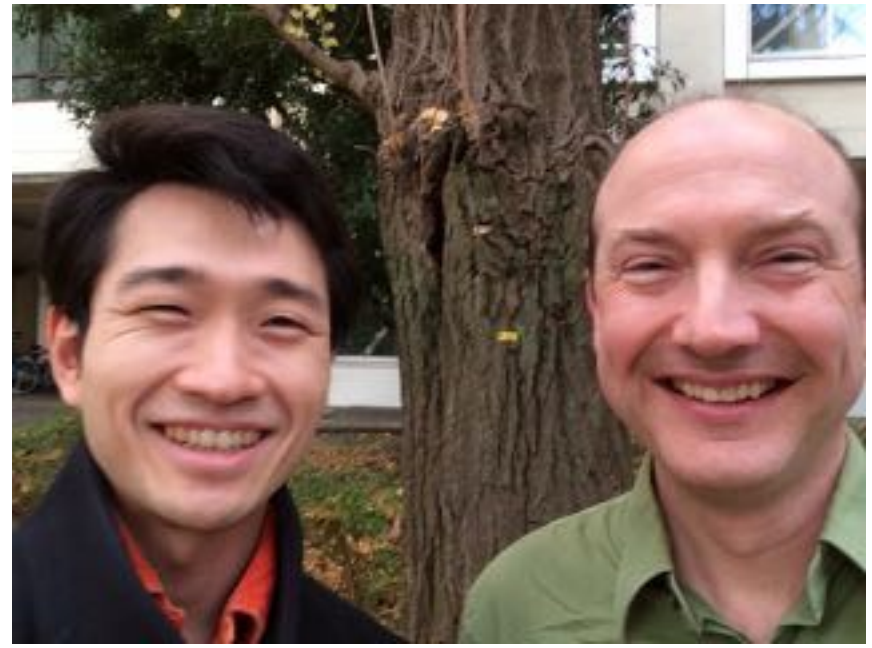
Tomohiro Tachi
The University of Tokyo



Thomas C. Hull
Western New England
University

Self-foldabiliy and Rigid Origami

Thomas Hull, Associate Professor of Mathematics
Western New England University
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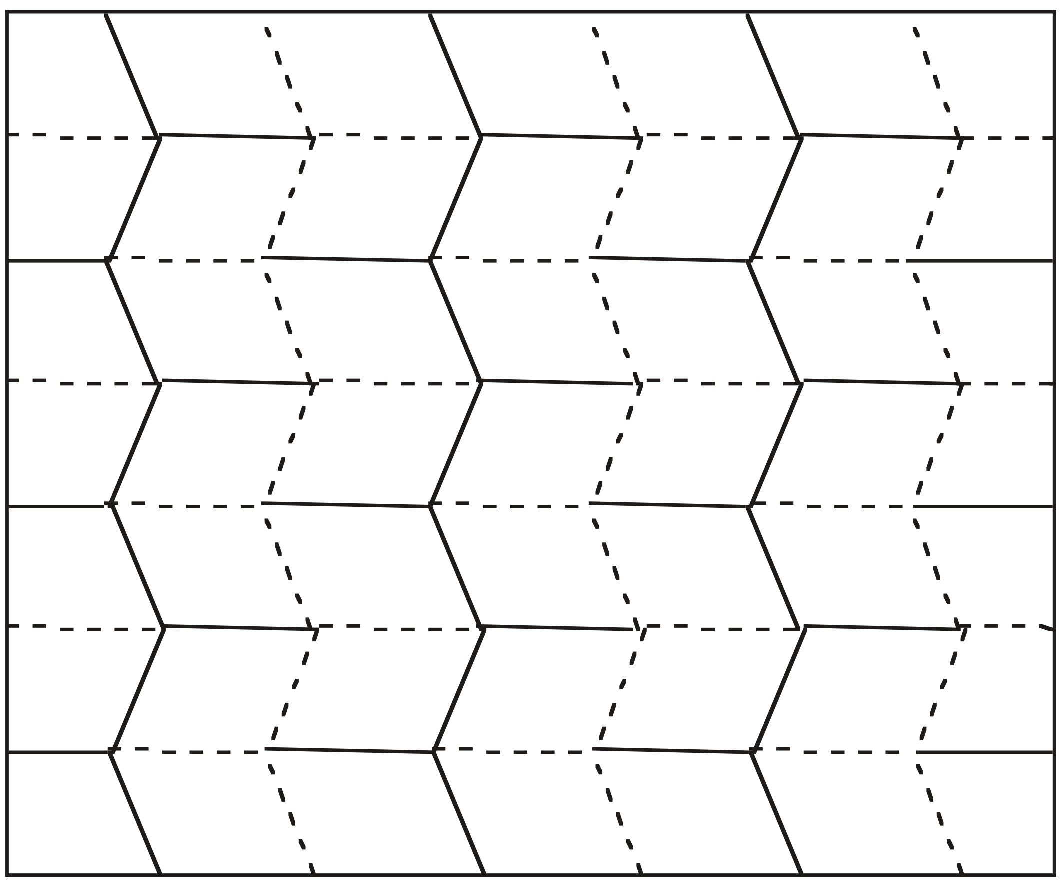
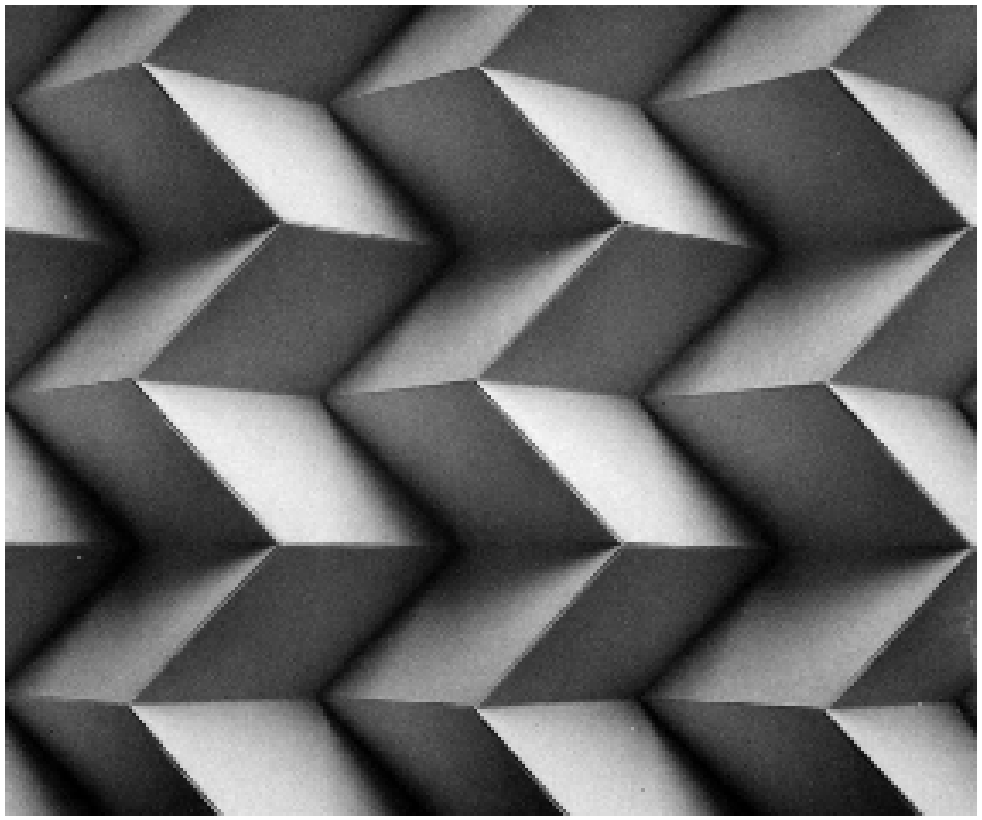
supported by: NSF grant EFRI-1240441
Mechanical Meta-Materials from Self-Folding Polymer Sheets

What is origami good for?

- The Miura map fold, invented by Japanese astrophysicist Koryo Miura in the 1970s, has been used for maps, solar panels in space satellites, and in nature.

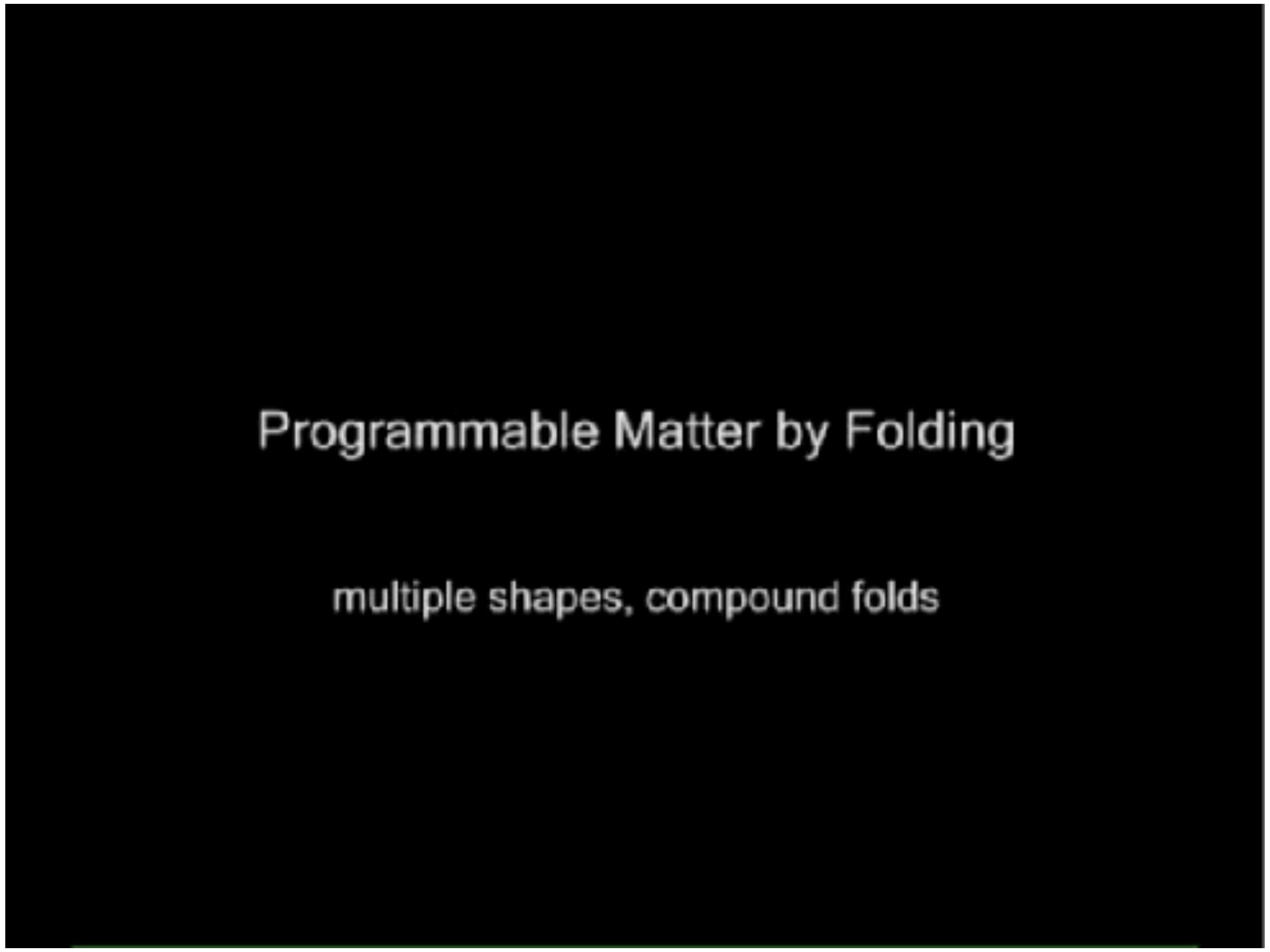




What is self-folding?

- Devising ways to make materials fold automatically in response to some stimulus.

- Example: Harvard Microrobotics Lab (Hawkes, An, Benbernou, Tanaka, Kim, Demaine, Rus, Wood 2009)



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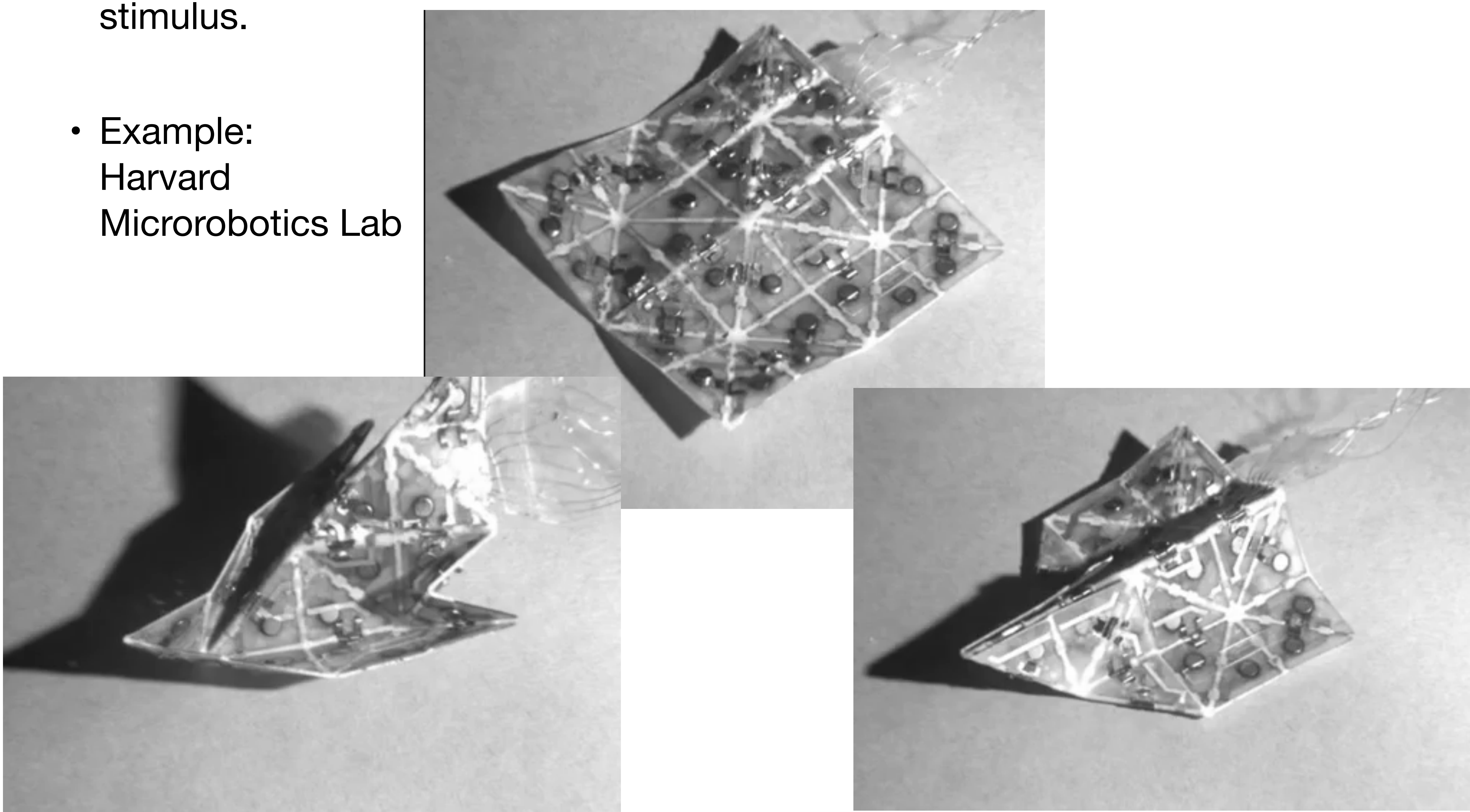


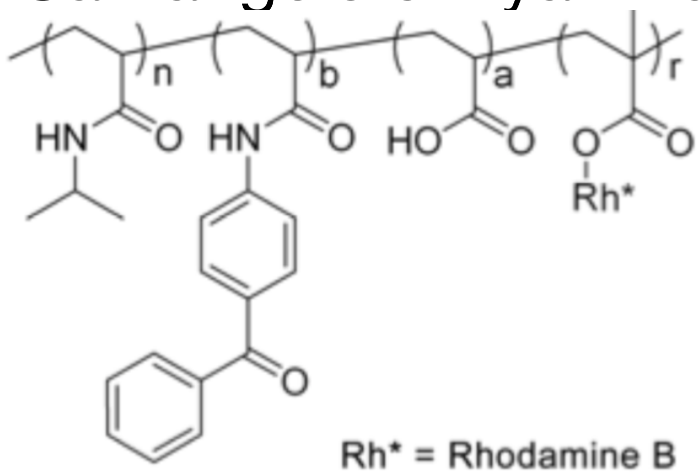
Robert Wood's robotics lab at Harvard (2014)

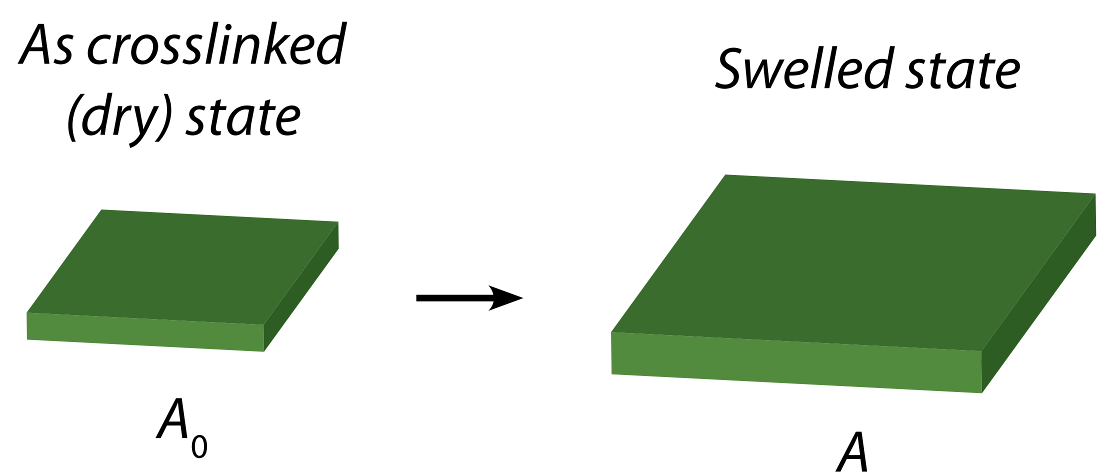
Larry Howell's group (BYU, 2014)

Larry Howell’s group (BYU, 2014)

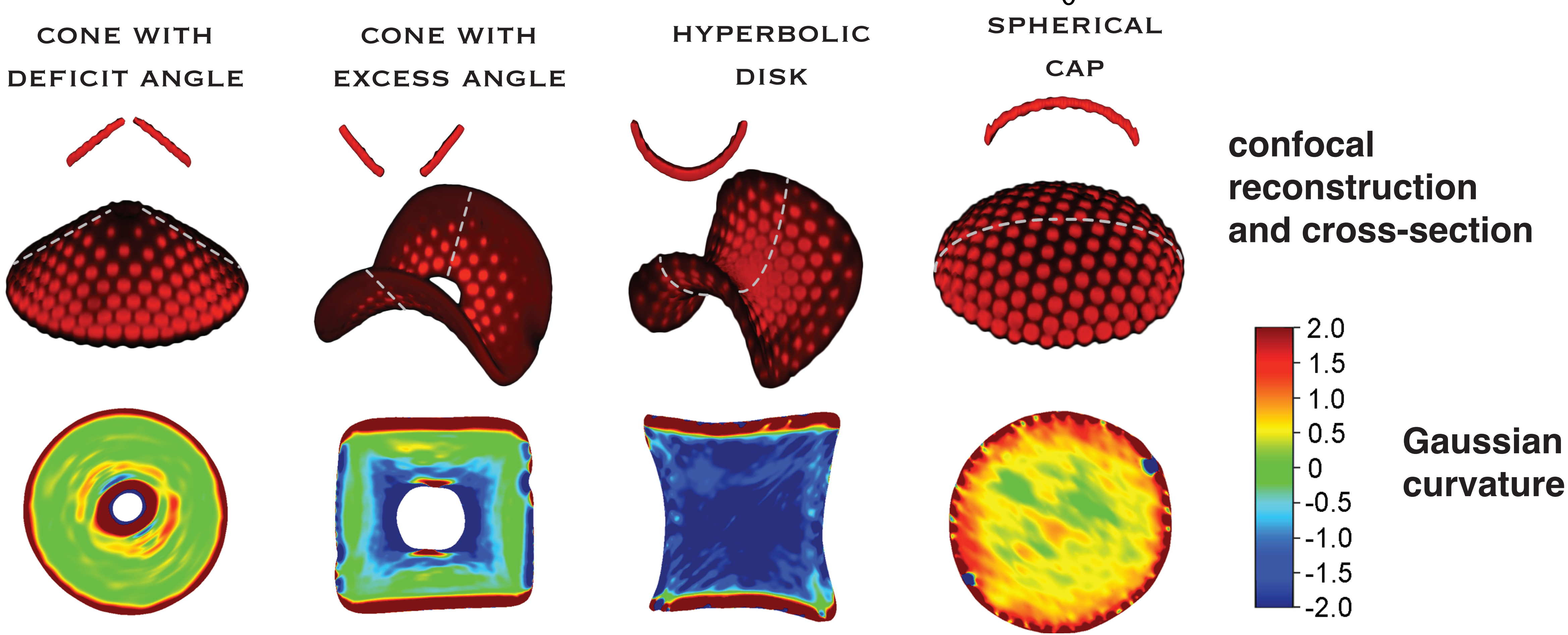
UMass Soft Matter & Polymer Science Group

- Chris Santangelo & Ryan Hayward



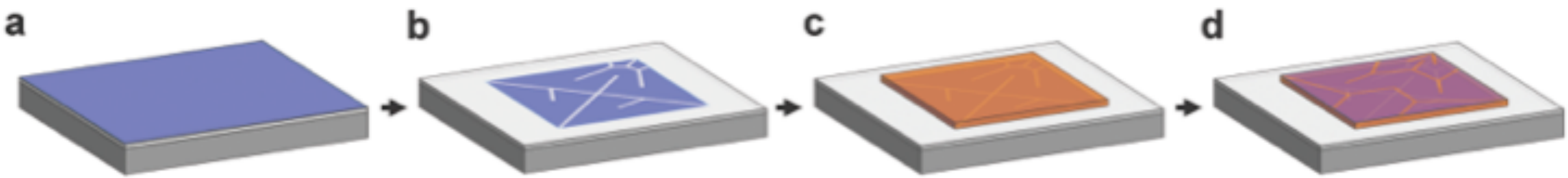


$$\Omega = A/A_0$$

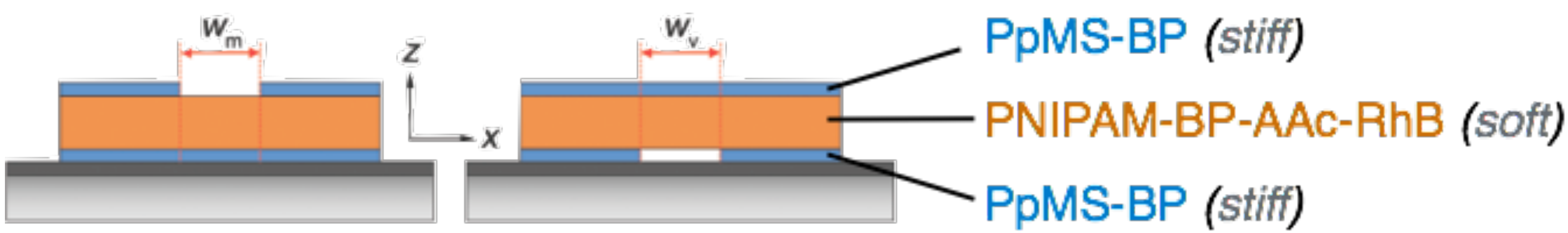


Self-folding polymer gels

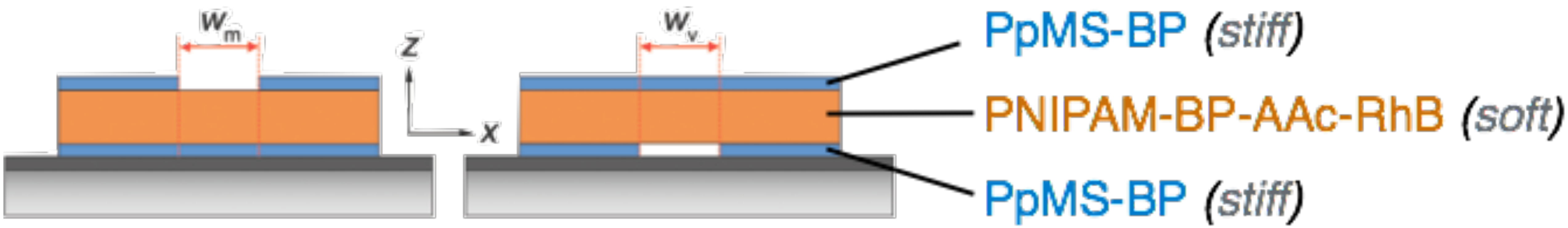
Three-step lithographic patterning of trilayer gels:

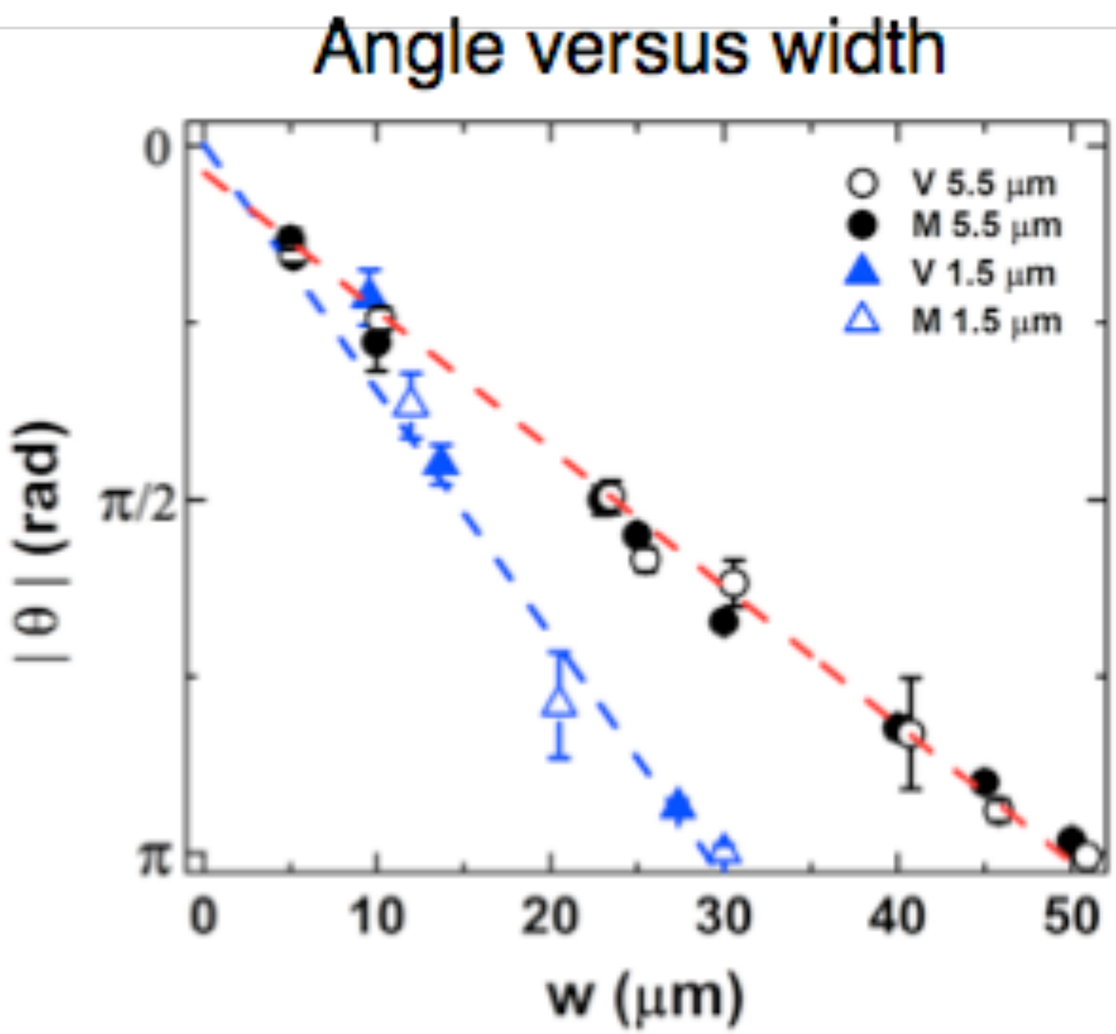


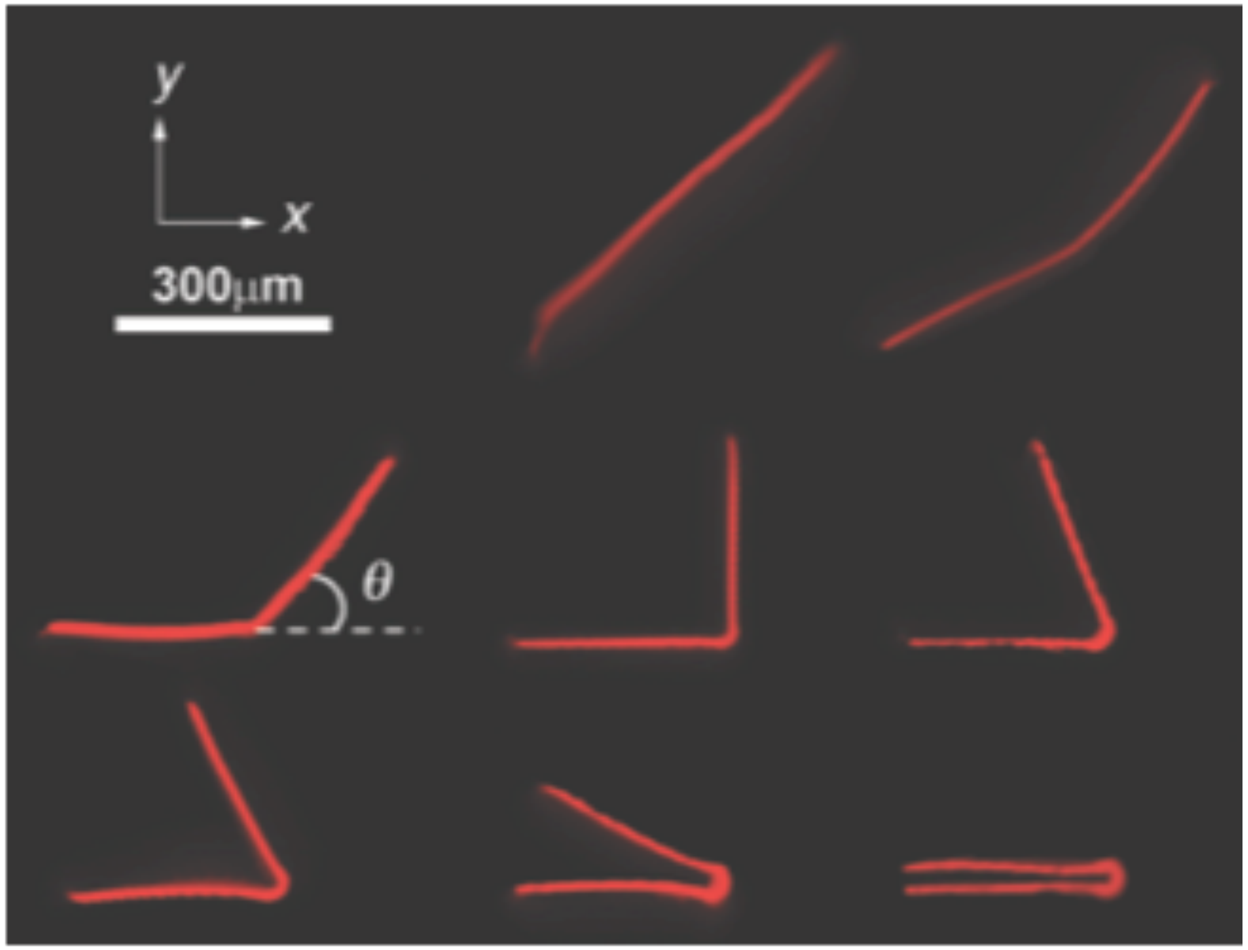
Schematic side-view of a fold:



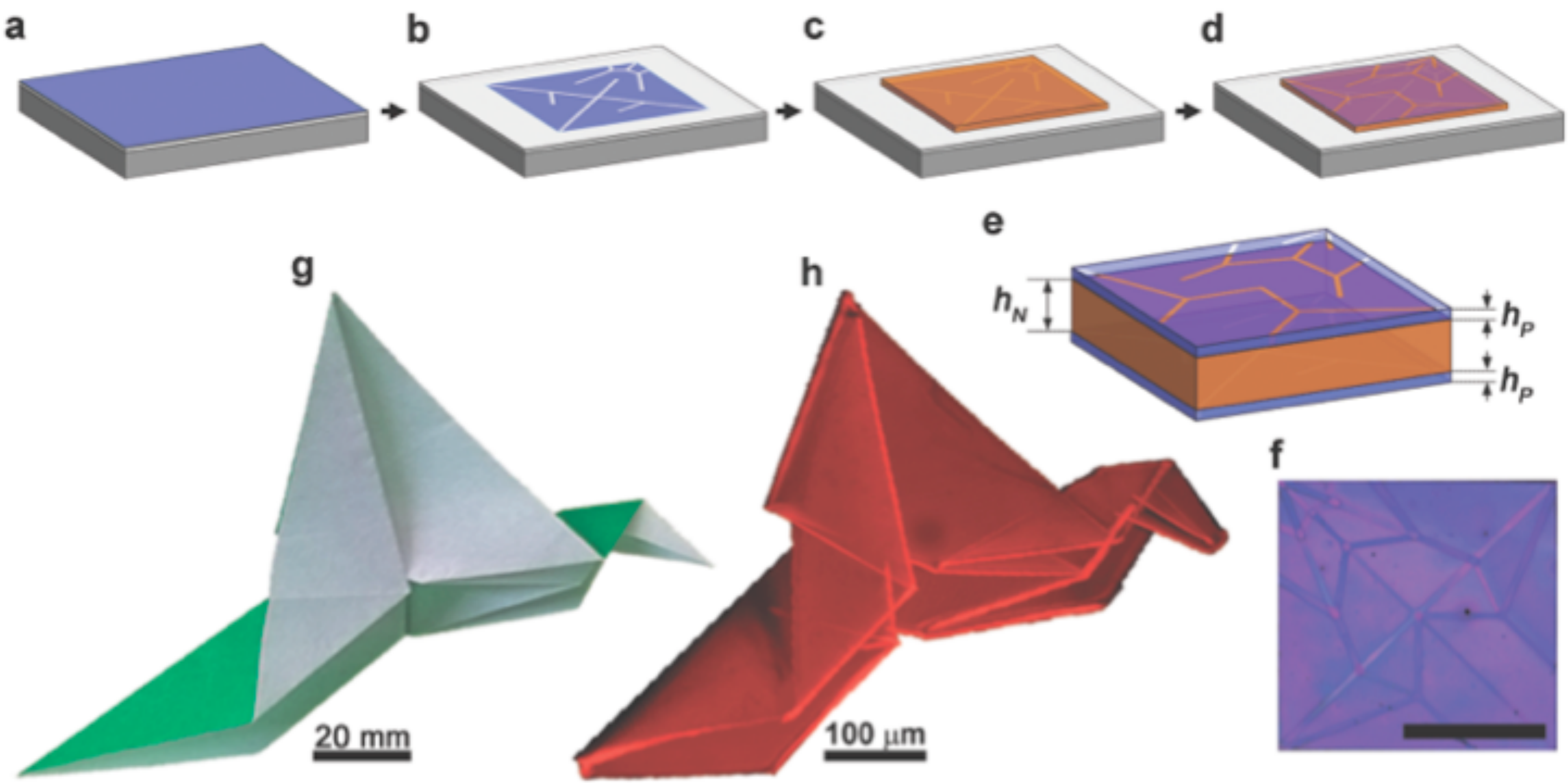
Self-folding polymer gels





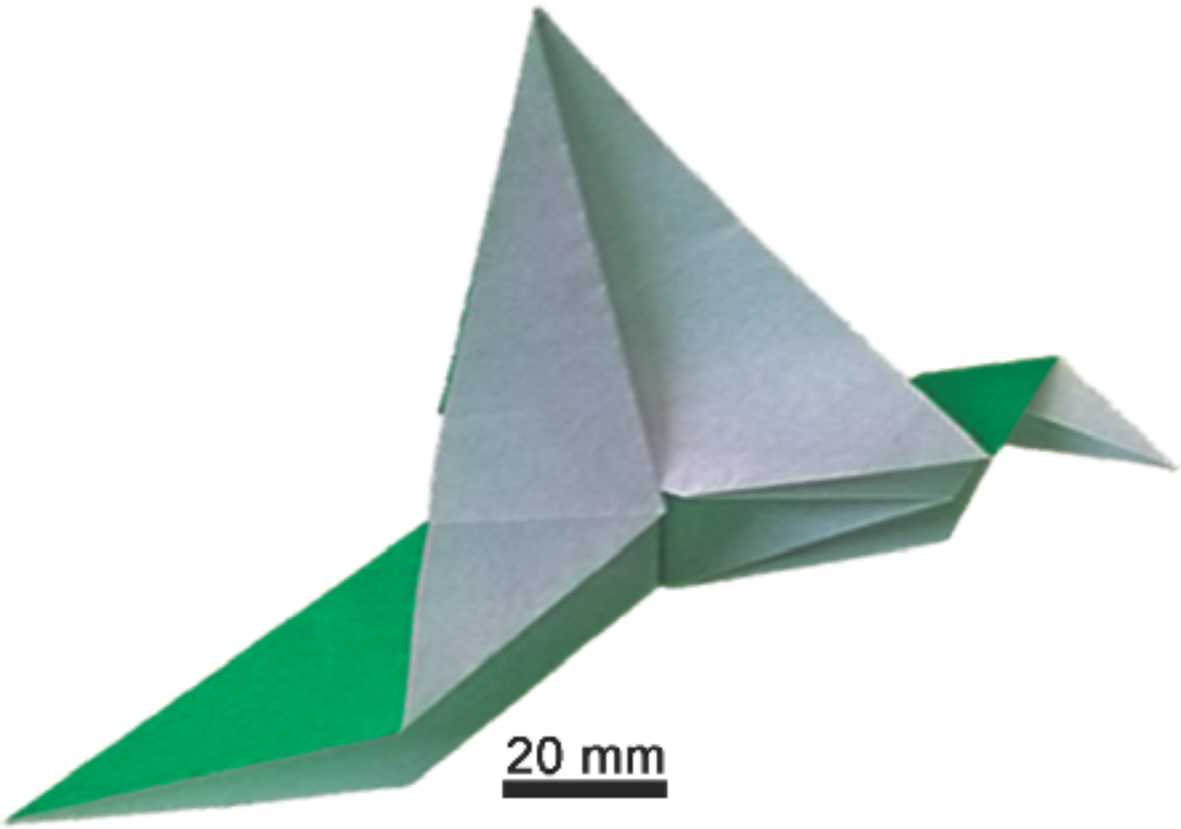


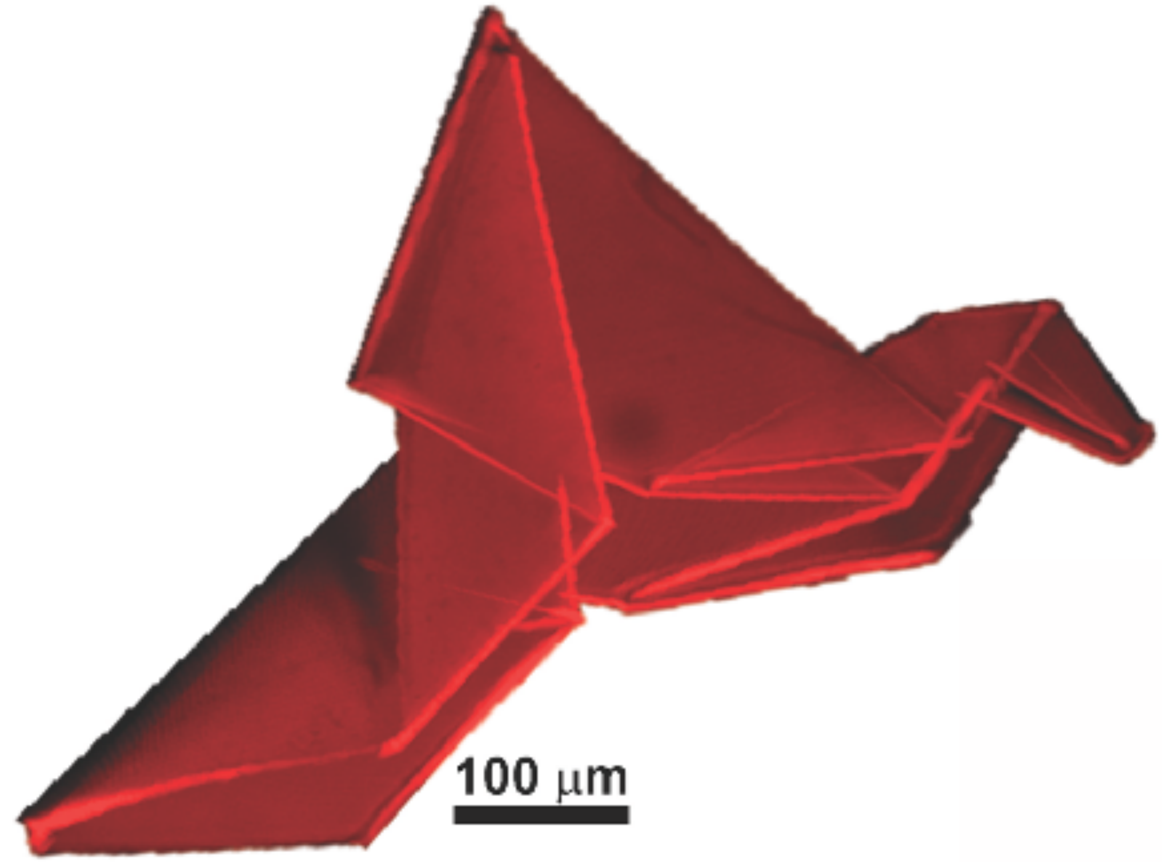
Can make origami with simultaneous folds



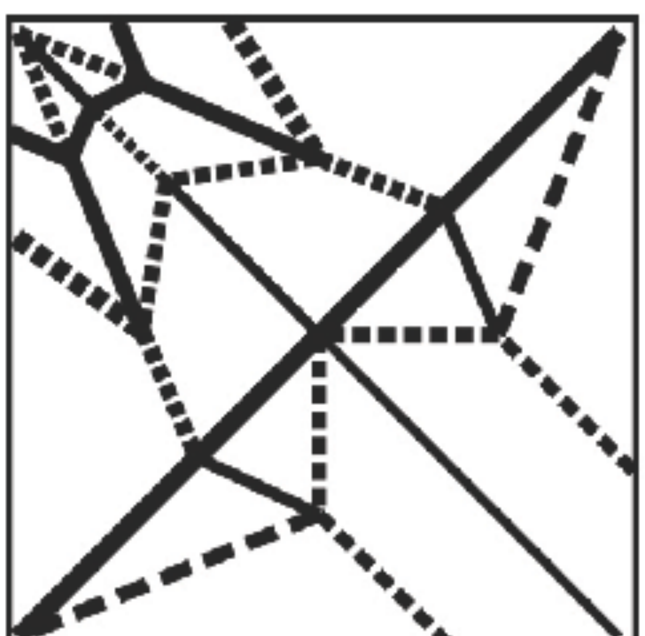
flapping Randlett bird

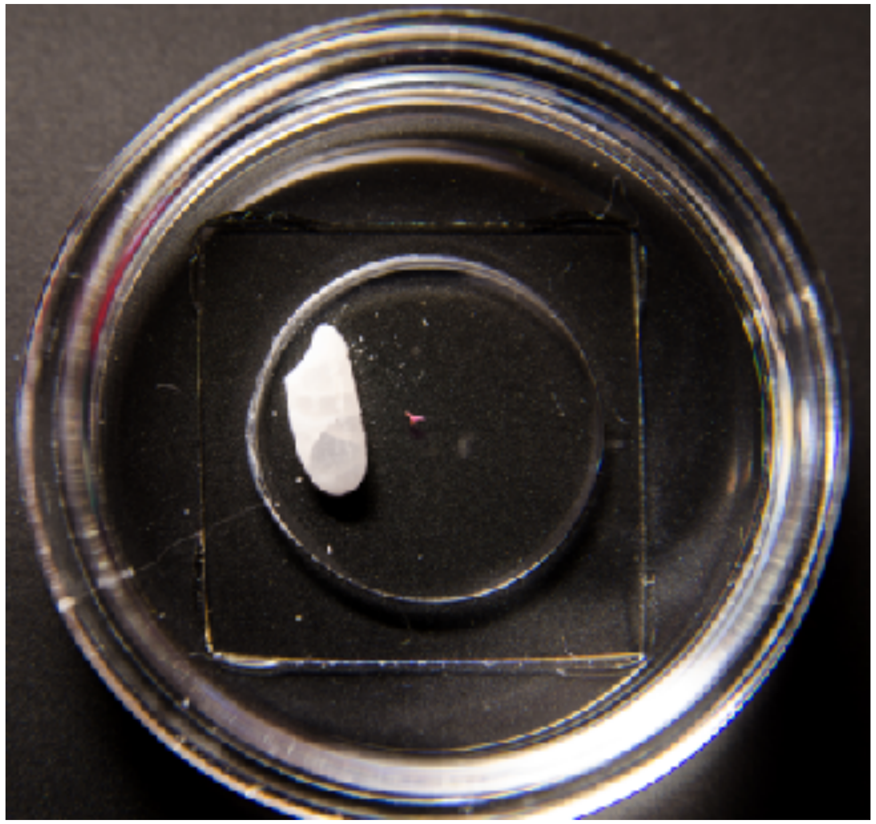
Can make origami with simultaneous folds



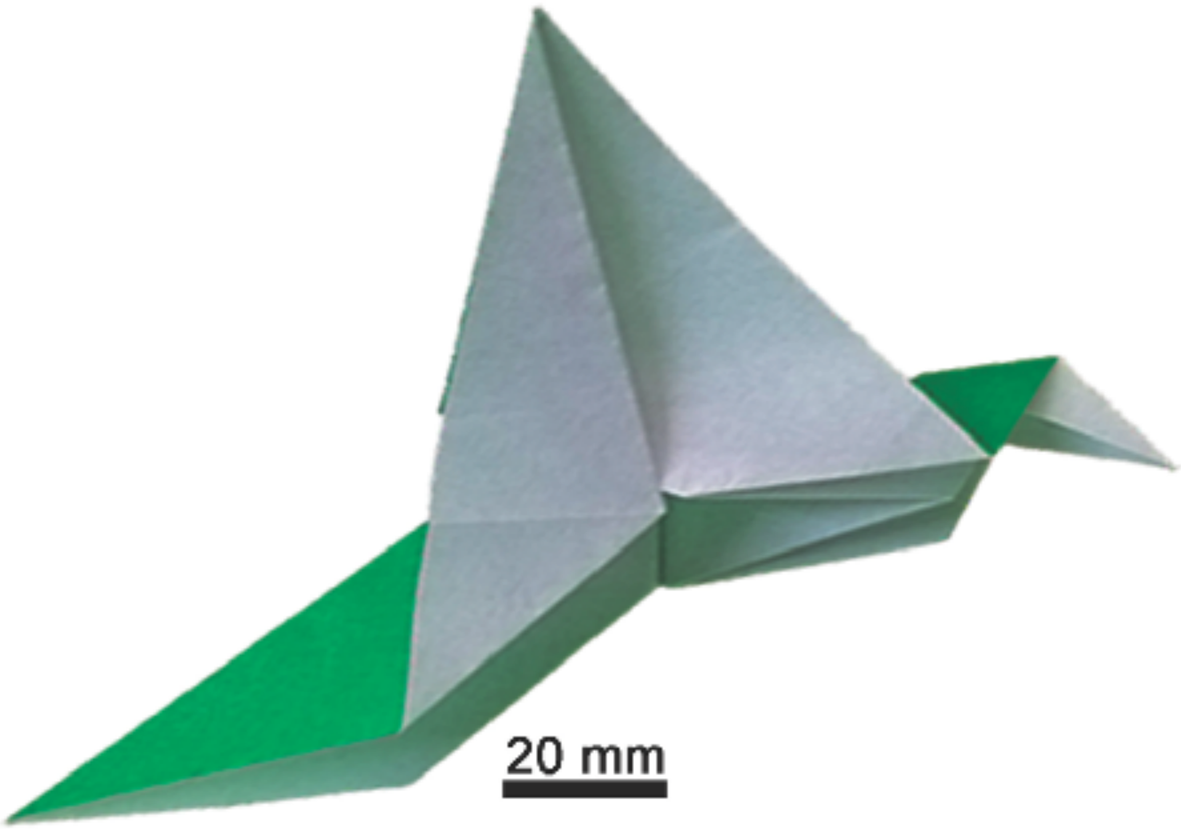


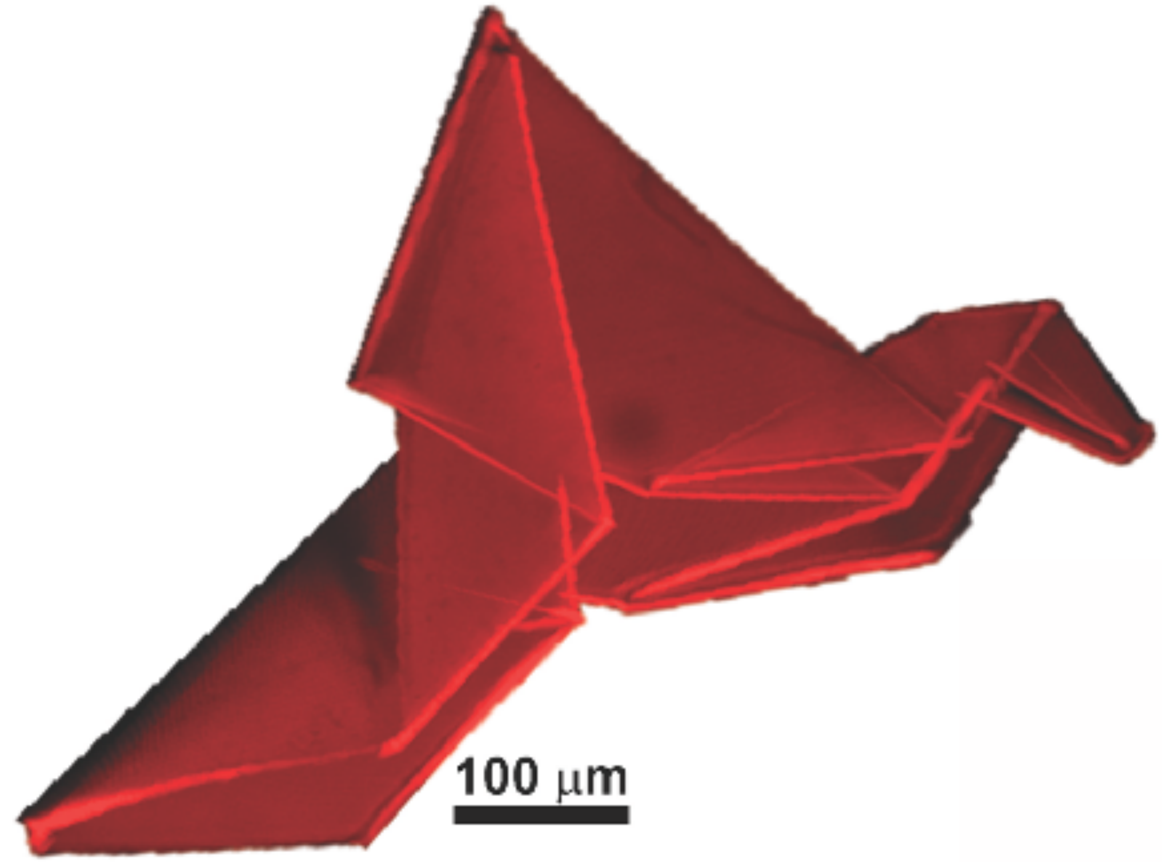
Randlett, “New Flapping Bird”



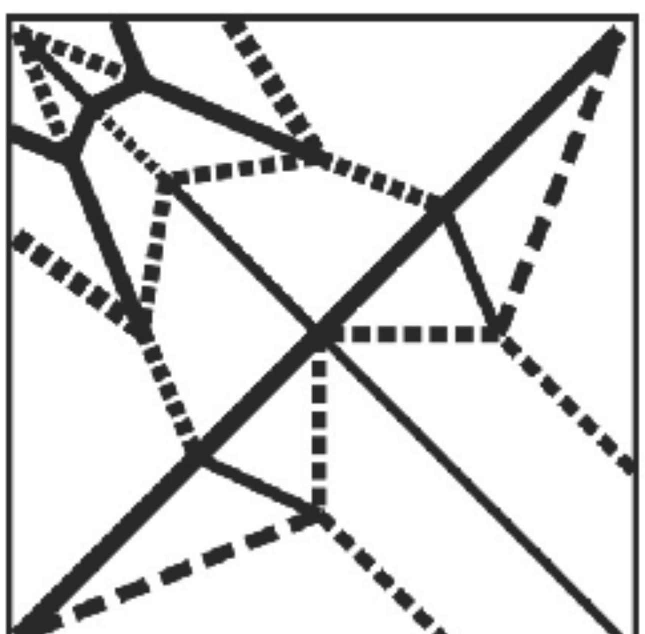


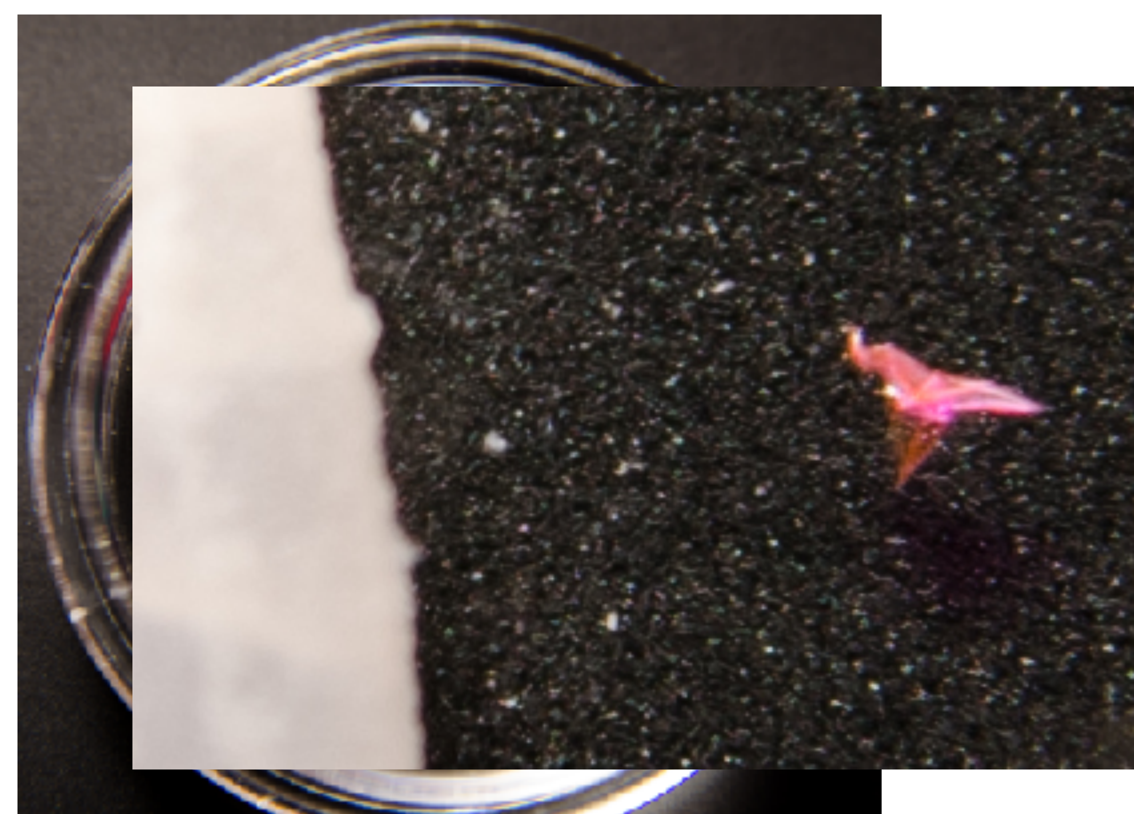
Can make origami with simultaneous folds

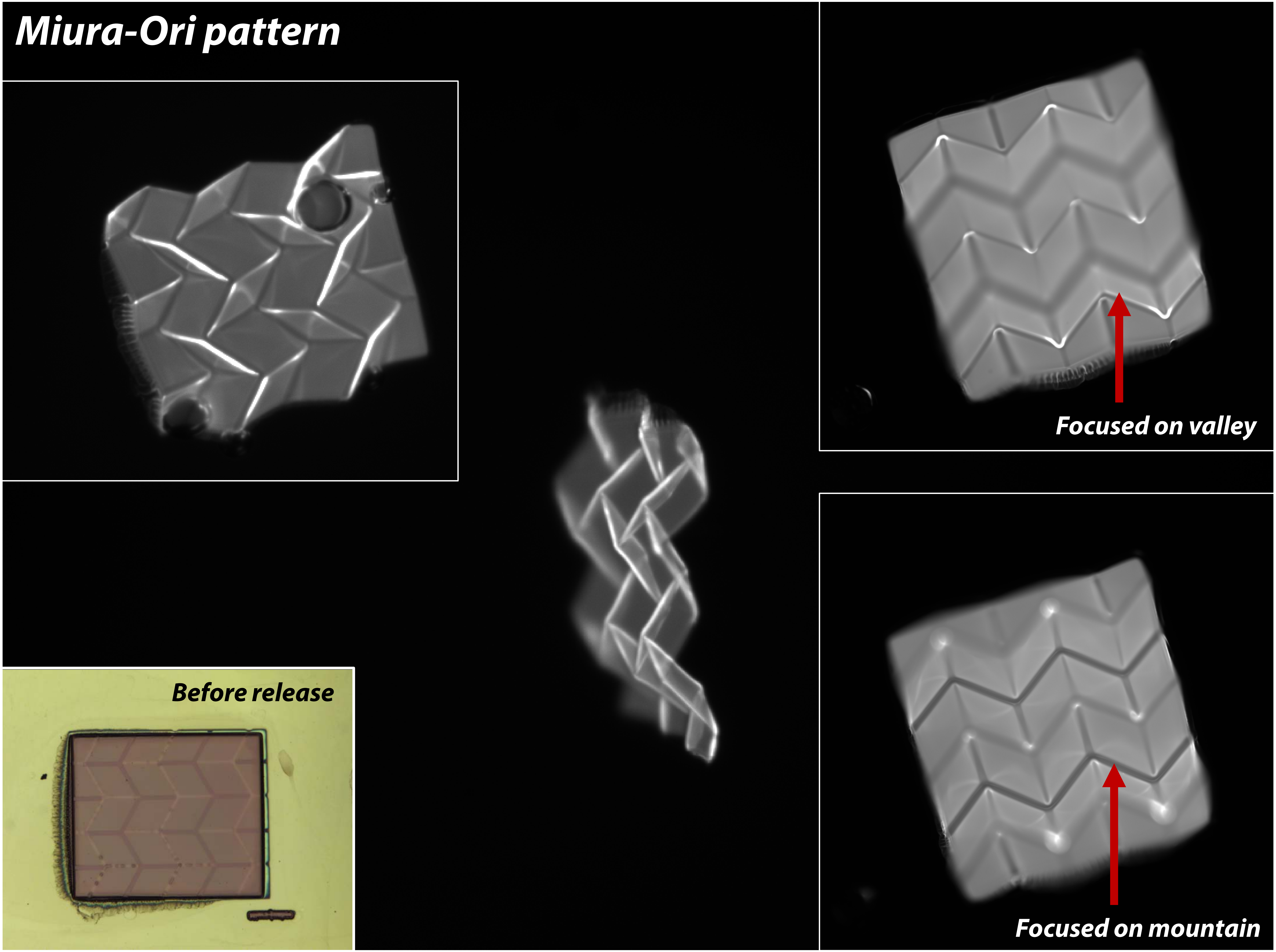




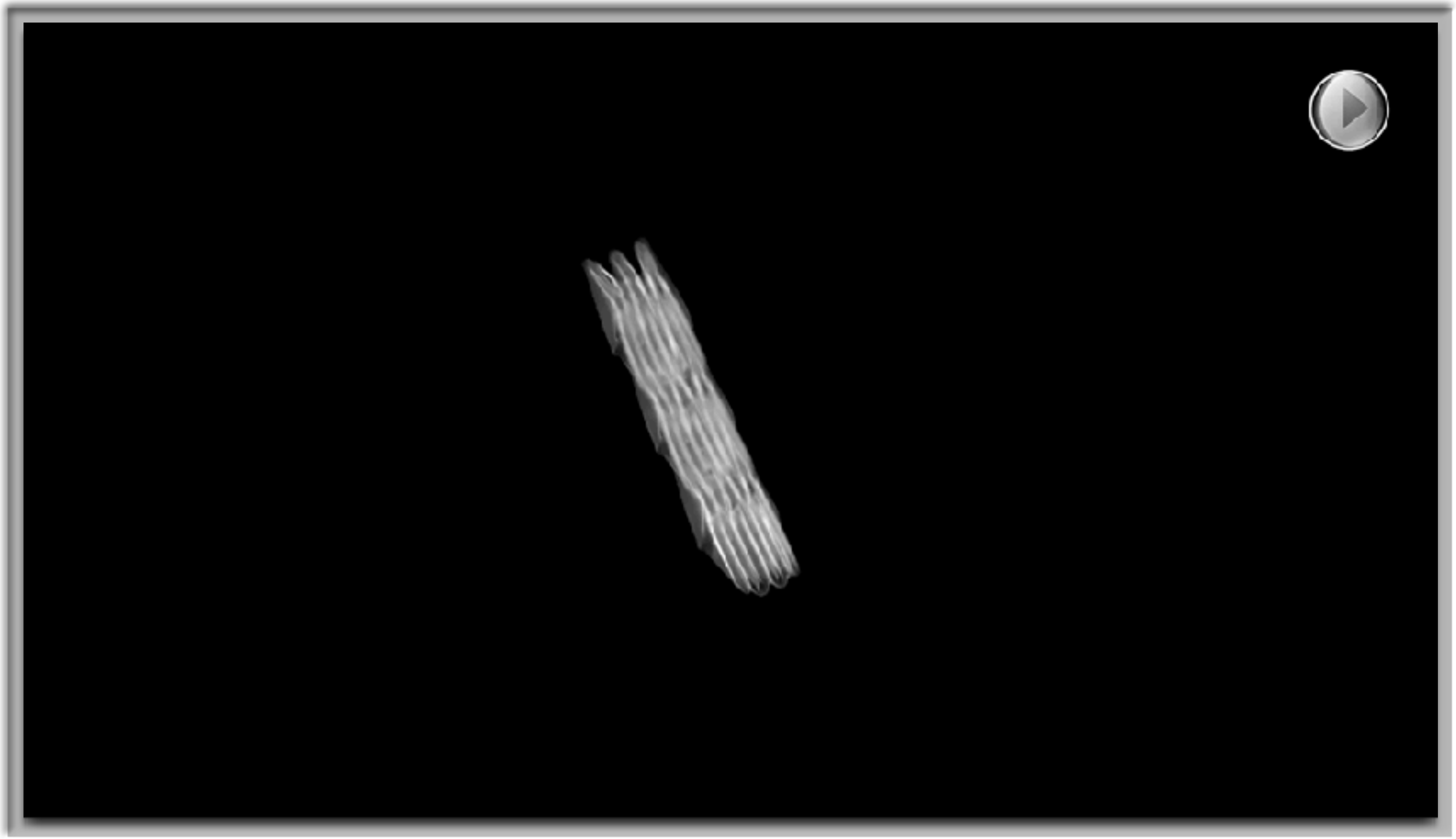
Randlett, “New Flapping Bird”






Miura-Ori pattern
Before release
Focused on valley
Focused on mountain

Swelling and Deswelling
(Na, Evans, Bae, Chiappelli, Santangelo, Lang, Hull, Hayward, 2014)



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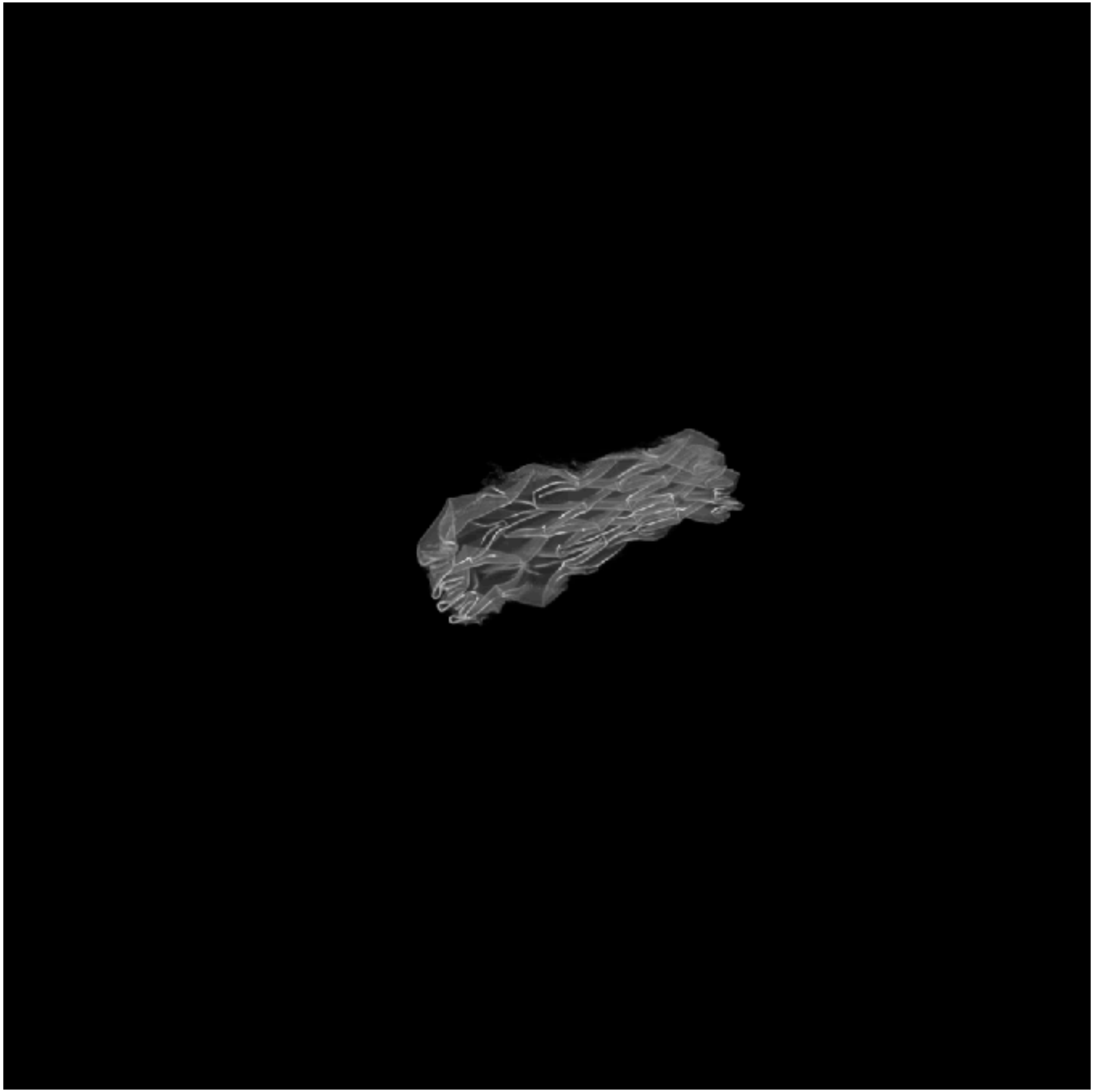

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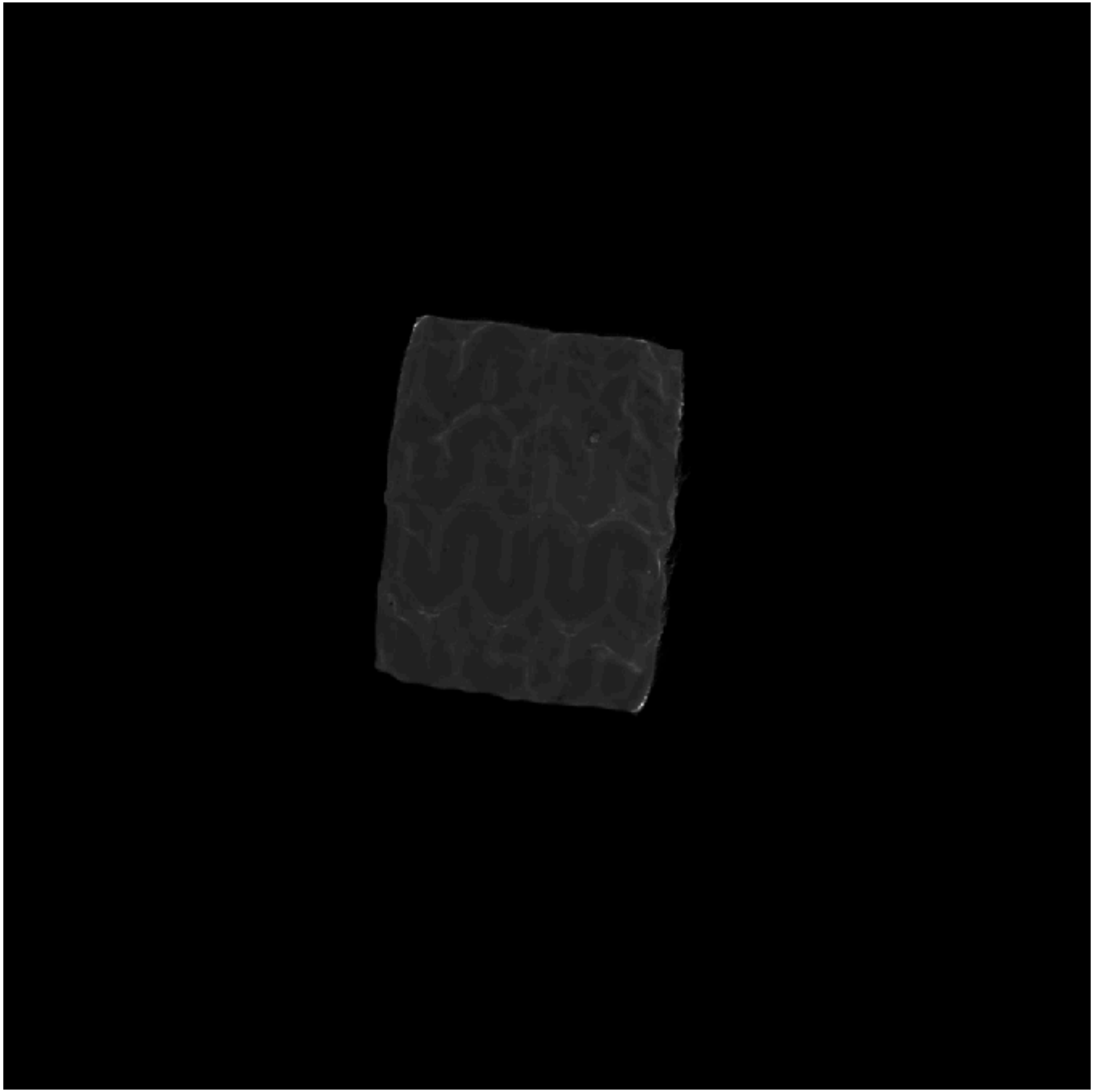

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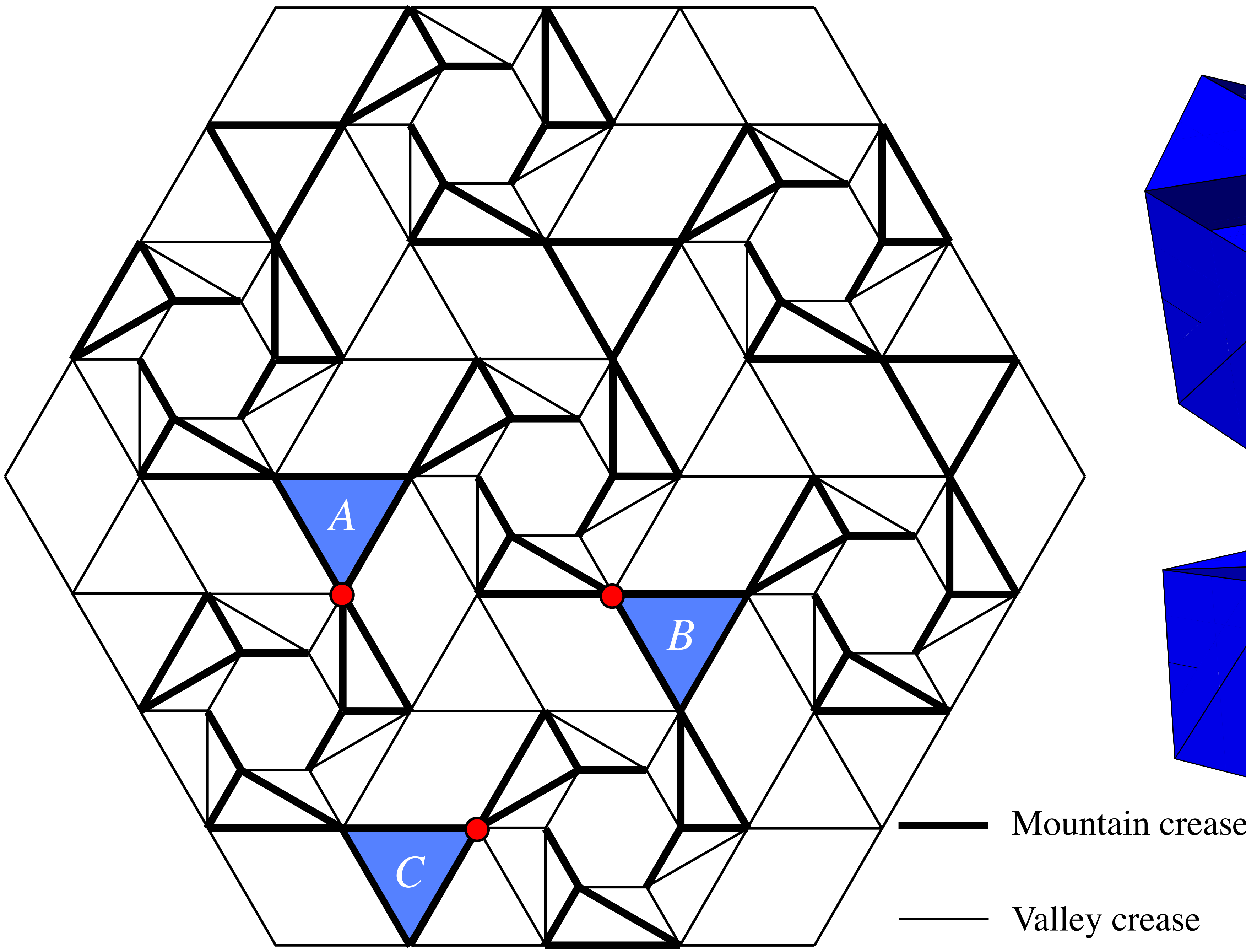
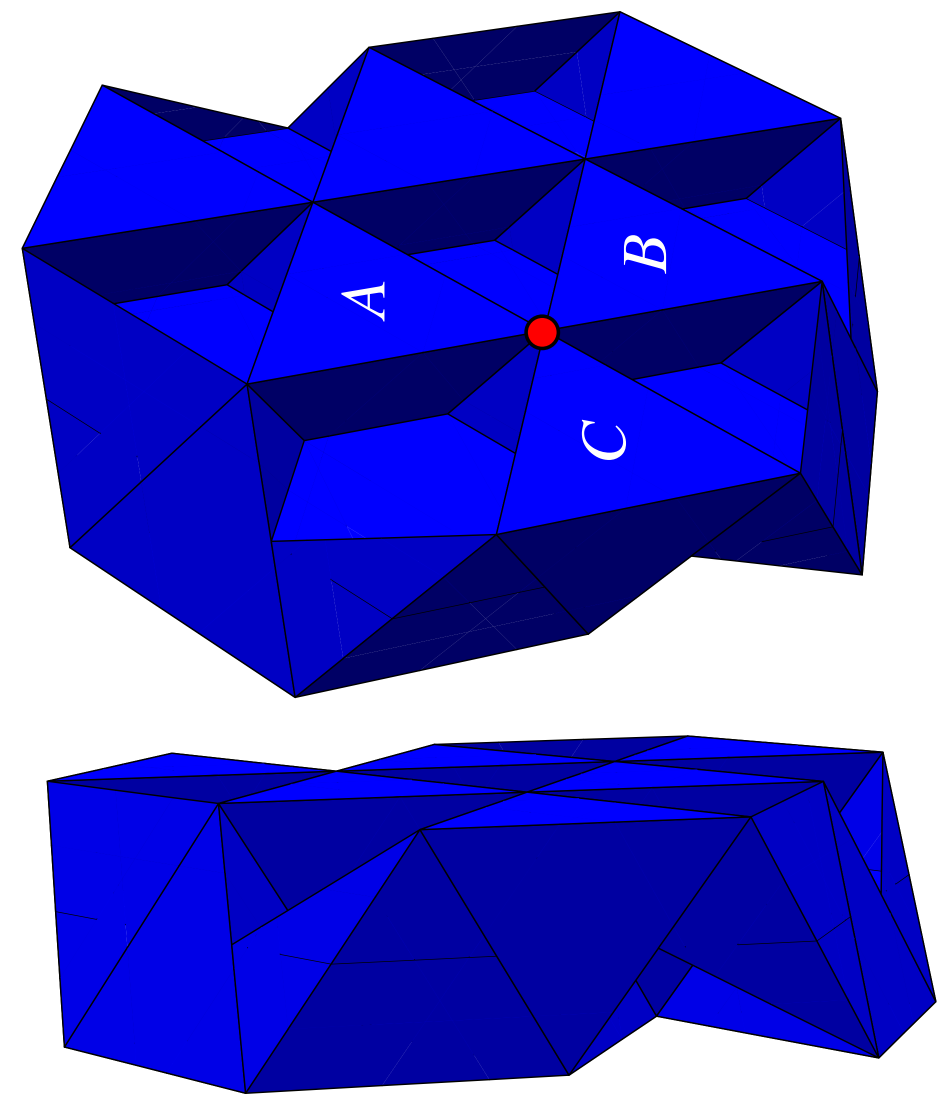
A more complicated fold

- Origami octahedron-tetrahedron truss (invented by numerous origamists: David Huffman, Ron Resch, Toshikazu Kawasaki, and others)





Octet Truss

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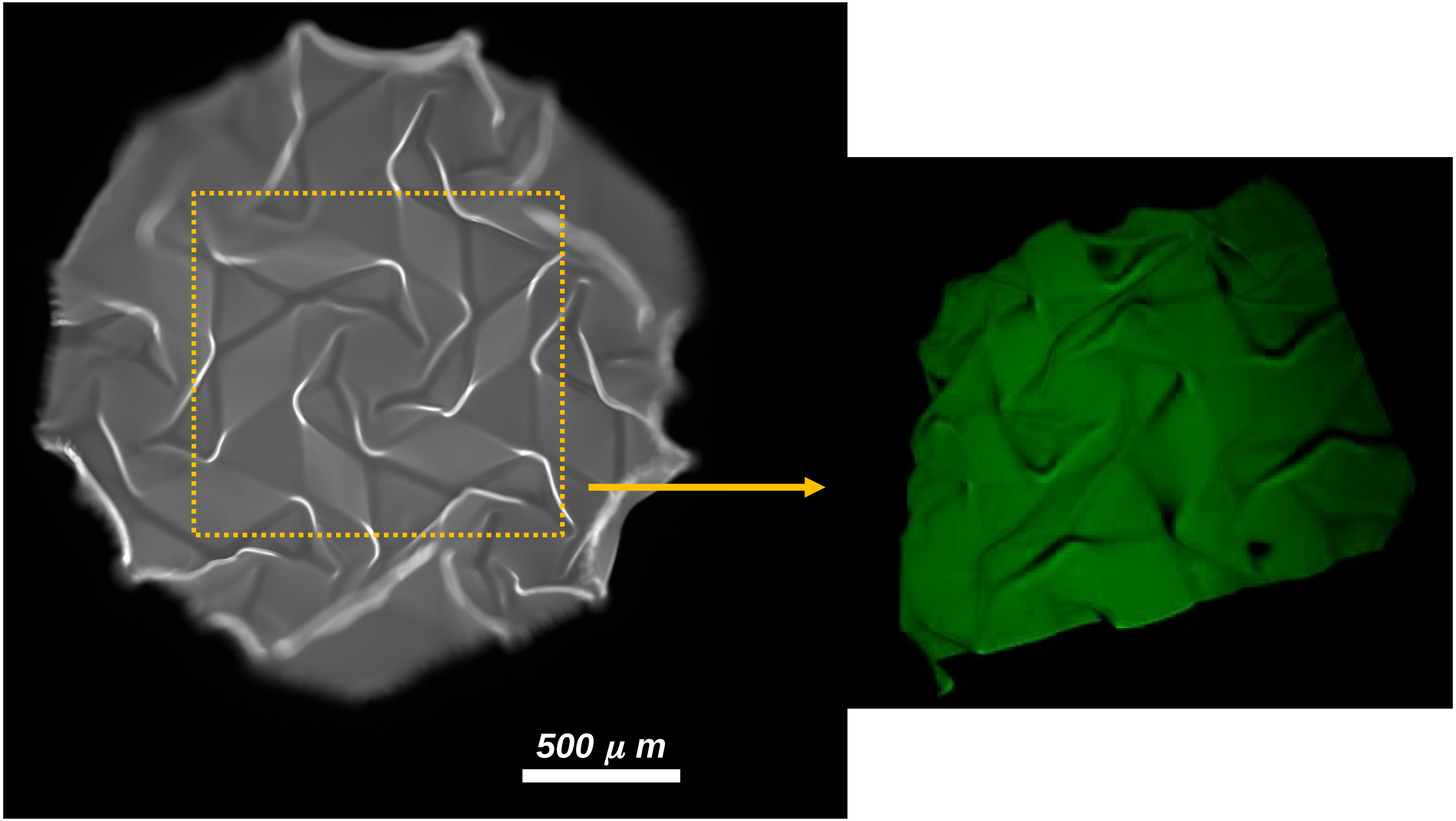
First attempts did not fold very well.



Octet Truss

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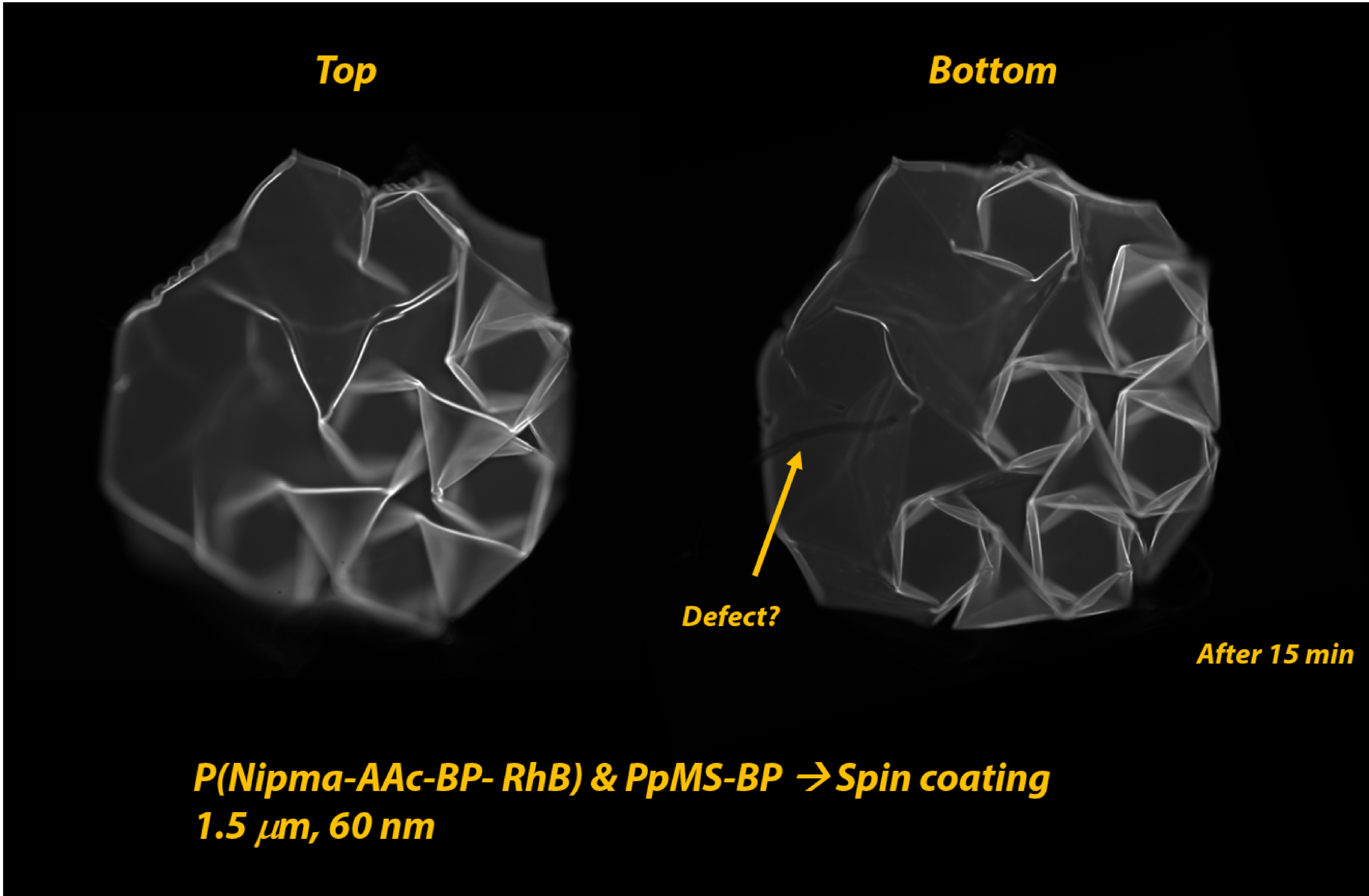
With more accurate folding angles, it worked much better.



Octet Truss

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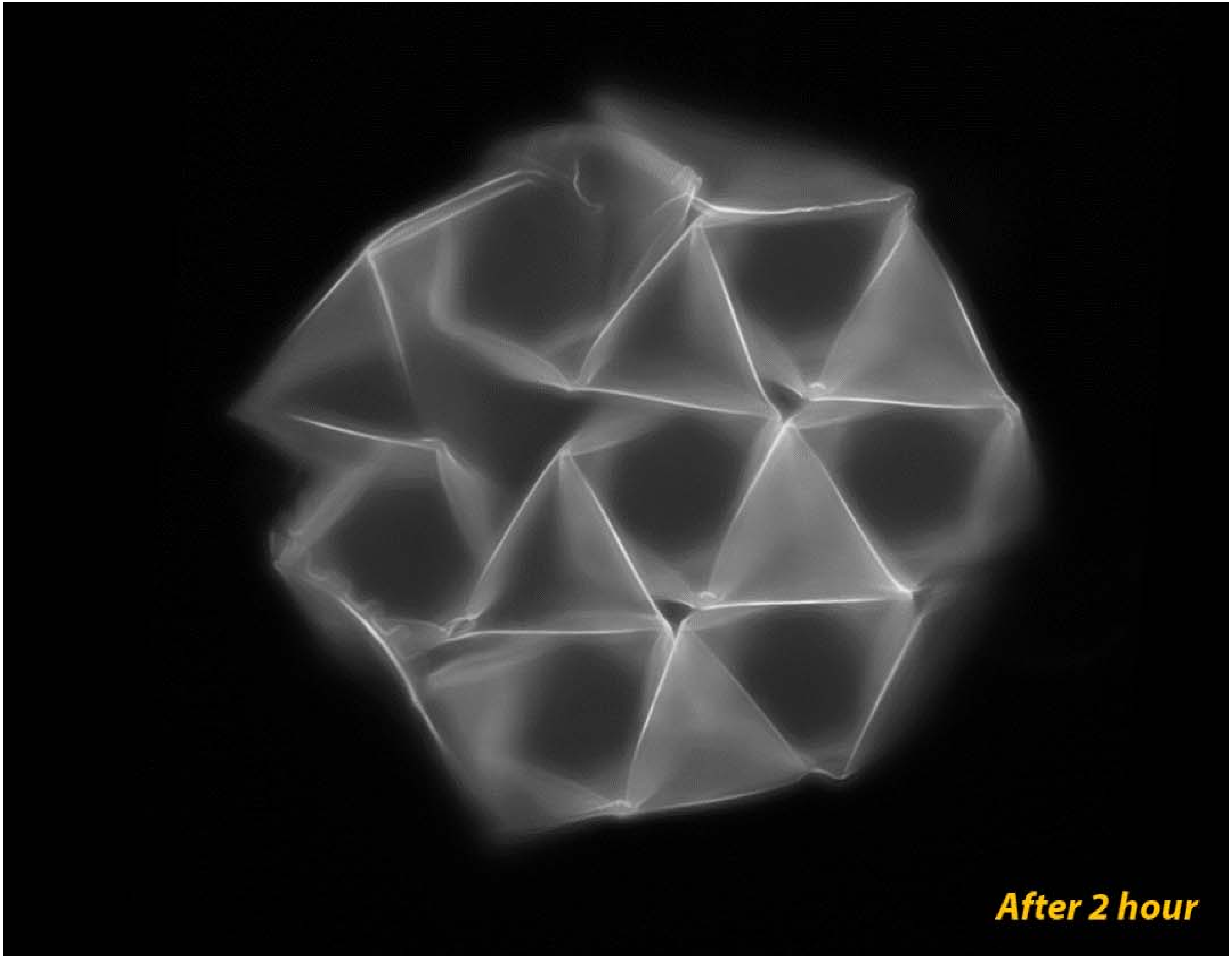

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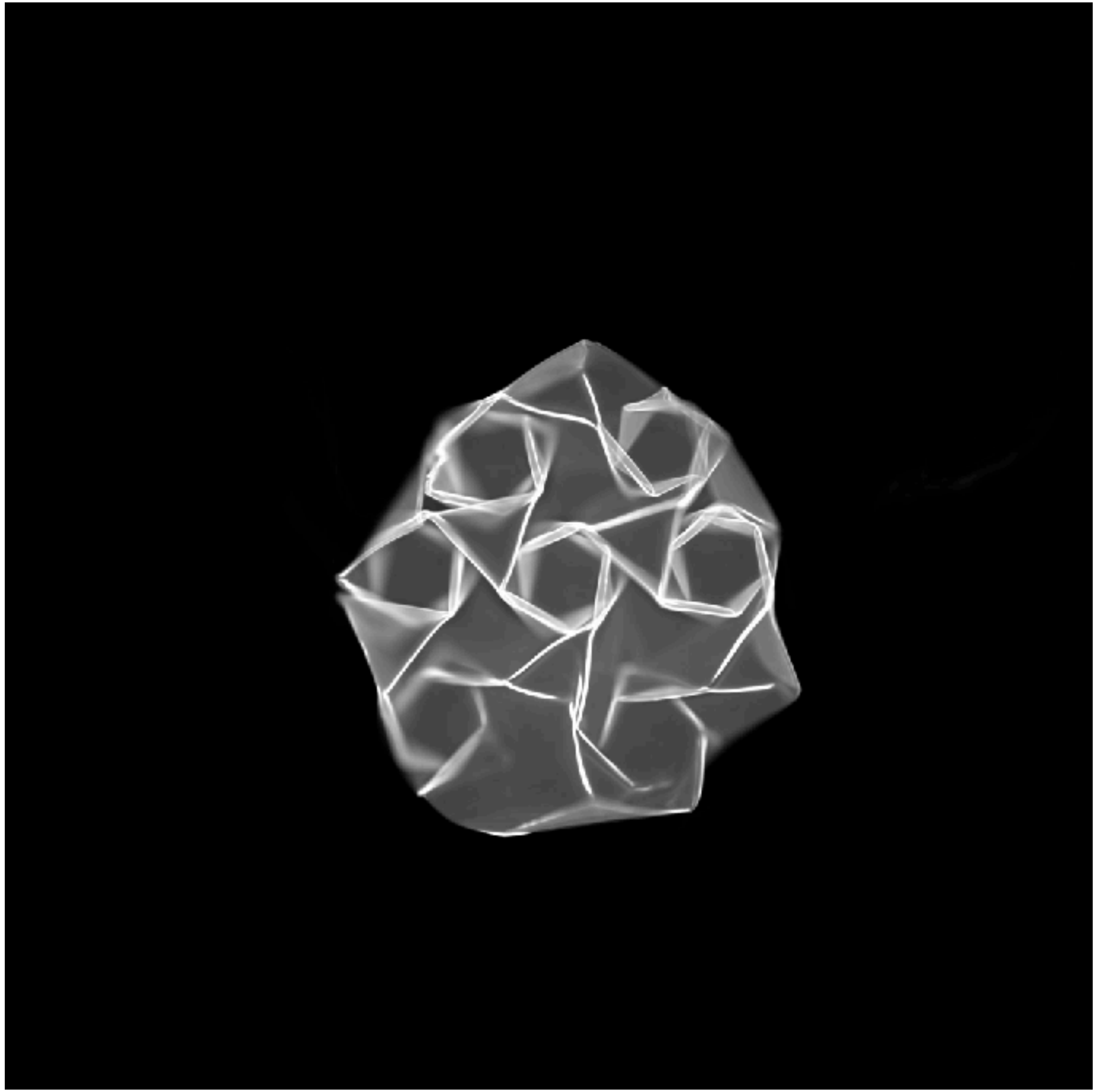

Octet Truss
(confocal fluorescence microscopy image by Junhee Na)

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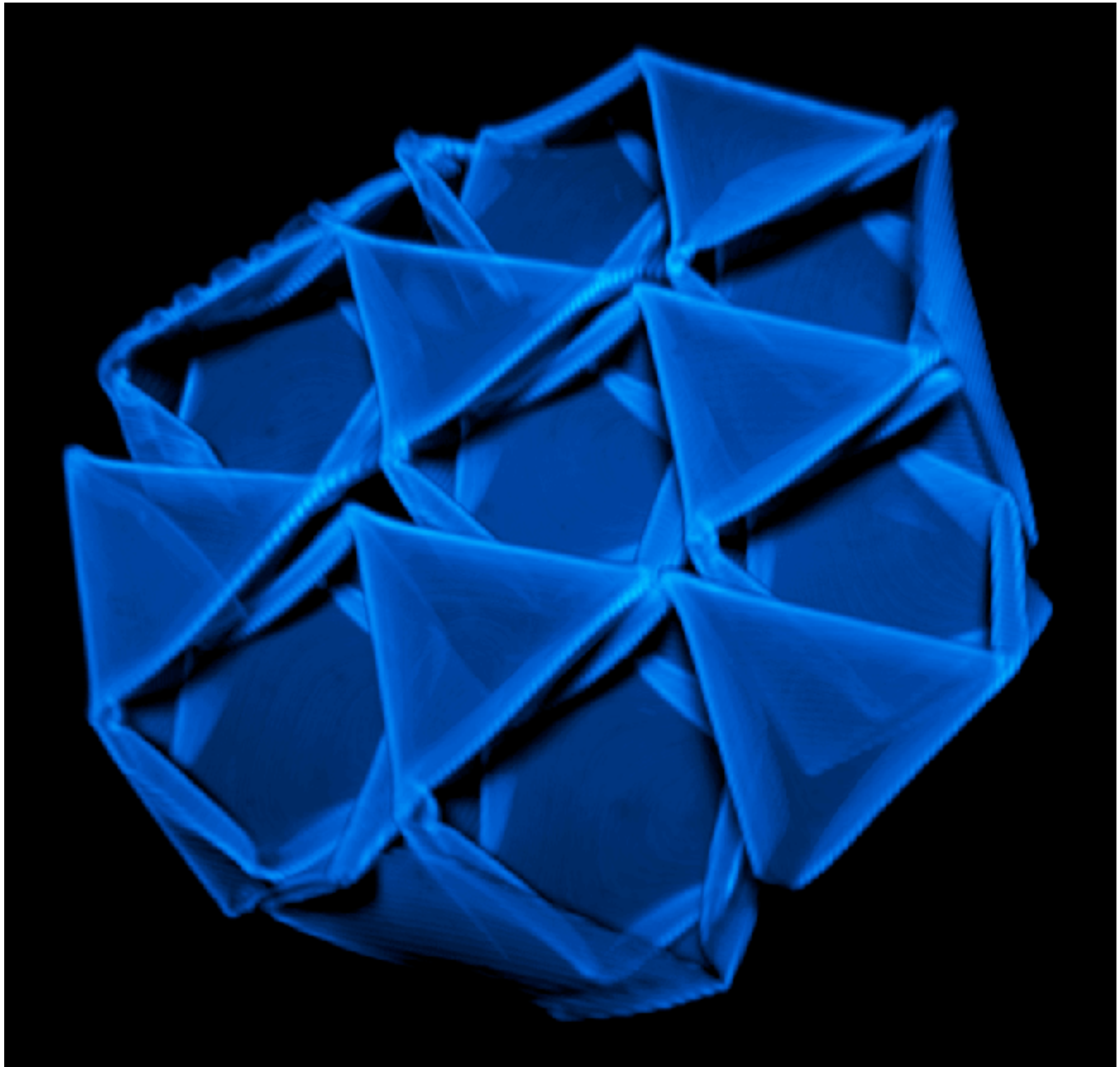
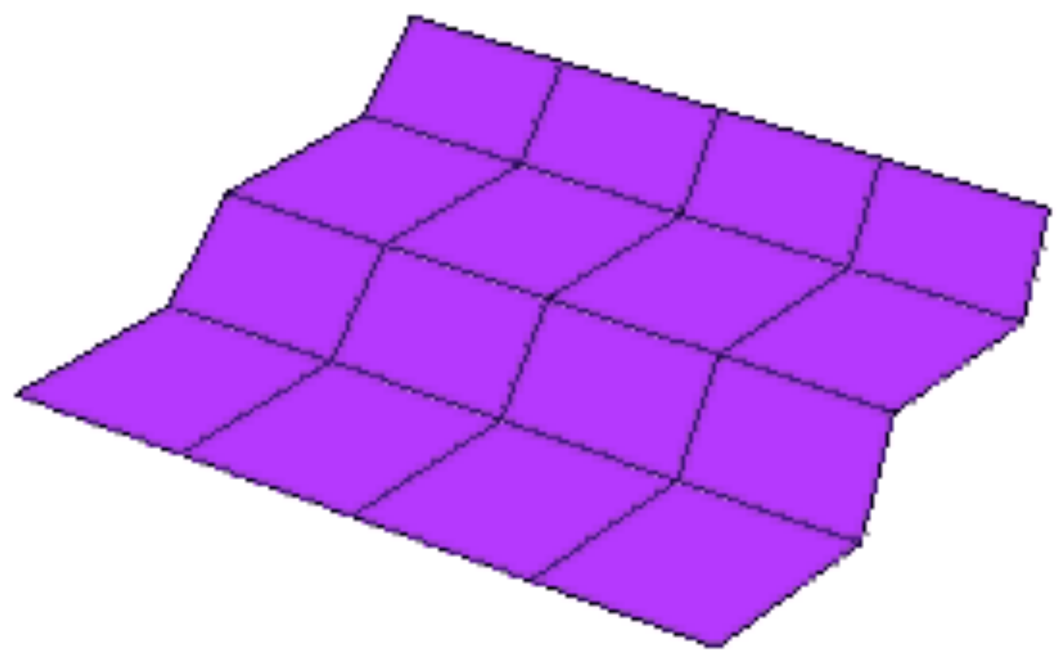


The classic way to fold a Miura-ori



How do we make such animations?

Configuration space of a flat-foldable, deg 4 vertex

Configuration space: n-dimensional parameter space containing points $\vec{\rho} = \langle \rho_1, \ldots, \rho_n \rangle$ where the ρ_i are all the folding angles will contain the **configuration space** containing all such points that satisfy the rigid folding constraints: Around every vertex we must have

$$R(l_1, \rho_1)R(l_2, \rho_2) \cdots R(l_{2n}, \rho_{2n}) = I$$

where $R(l_i, \rho_i)$ is the matrix that rotates 3D space about line l_i by ρ_i .

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Apply to this vertex:

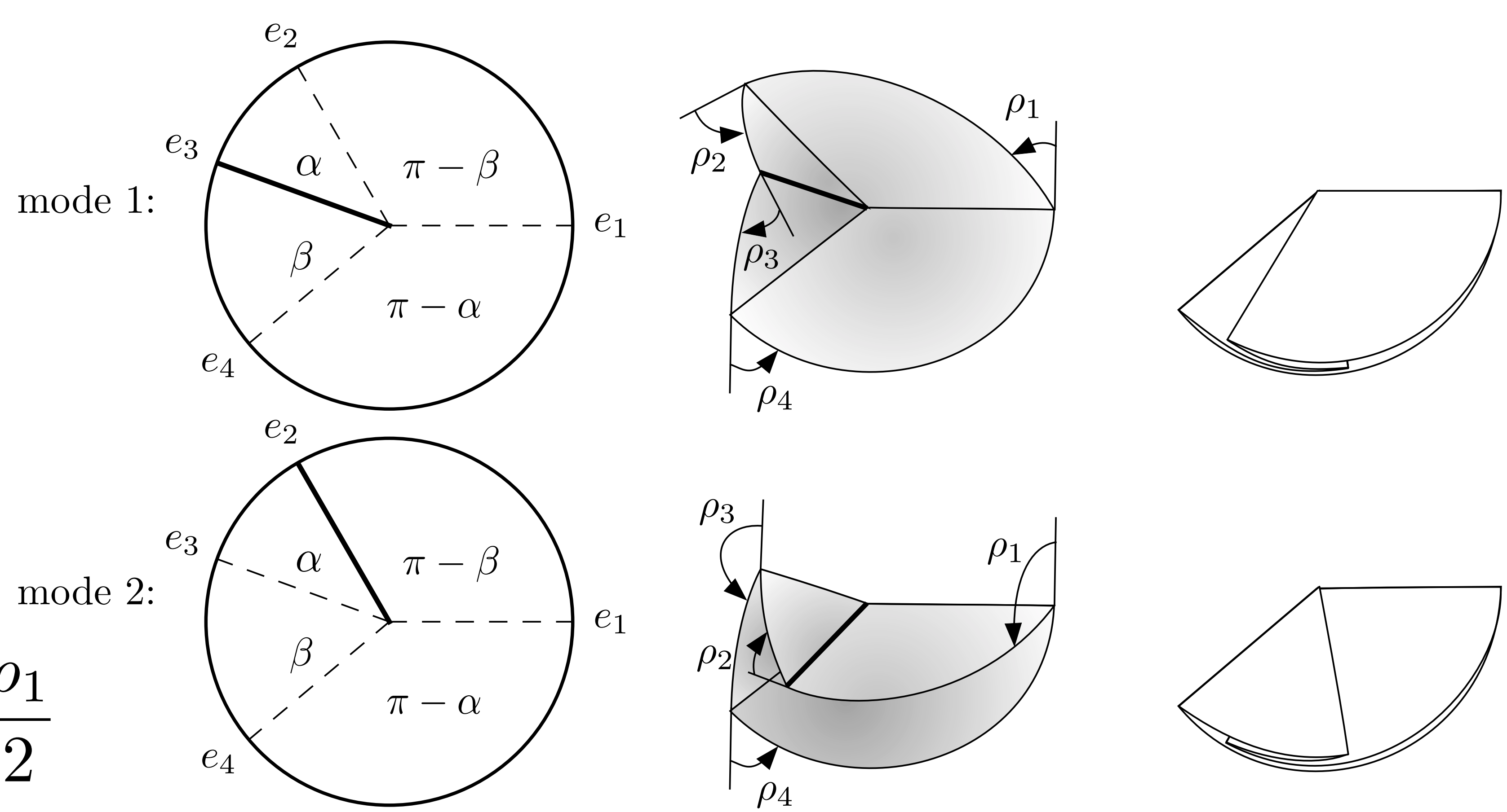
$$\Rightarrow \quad \rho_1 = -\rho_3, \rho_2 = \rho_4$$

$$\tan \frac{\rho_2}{2} = \frac{\cos(\frac{\alpha - \beta}{2})}{\cos(\frac{\alpha + \beta}{2})} \tan \frac{\rho_1}{2}$$

OR

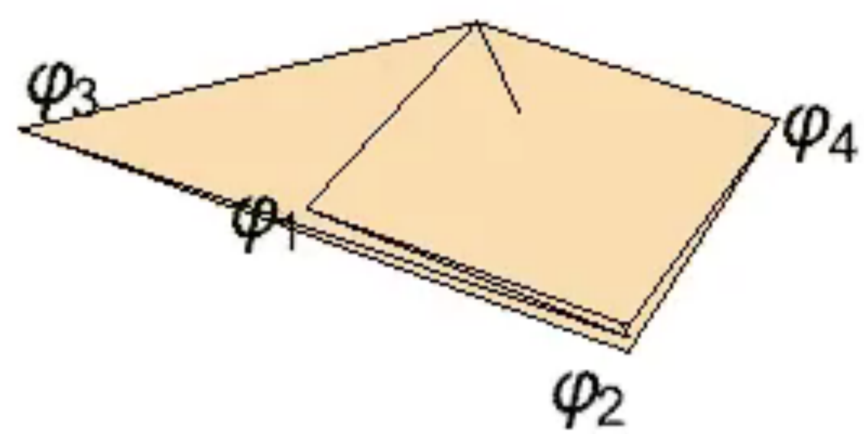
$$\rho_1 = \rho_3, \rho_2 = -\rho_4$$

$$\tan \frac{\rho_2}{2} = -\frac{\sin(\frac{\alpha - \beta}{2})}{\sin(\frac{\alpha + \beta}{2})} \tan \frac{\rho_1}{2}$$



Configuration space of a flat-foldable, deg 4 vertex

Demo:

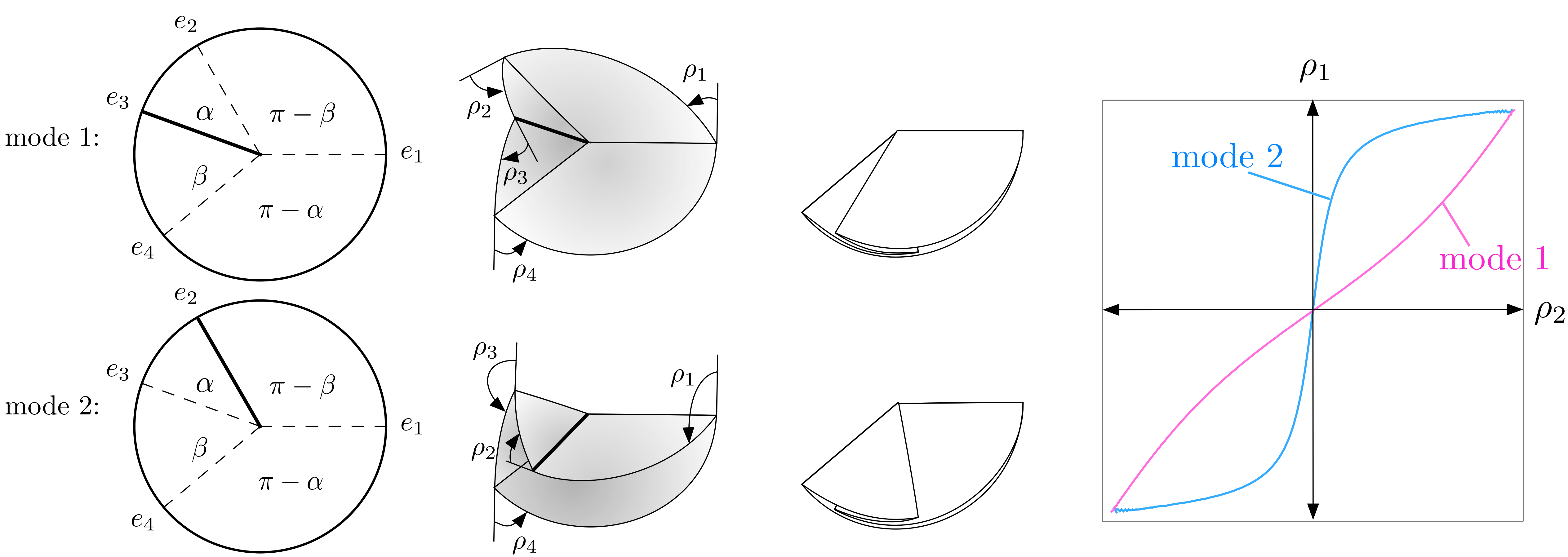


$\varphi_1 = -177.617 \quad \varphi_3 = -177.617$

$\varphi_2 = 174.251 \quad \varphi_4 = -174.251$

Configuration space of a flat-foldable, deg 4 vertex

What does the configuration space look like?



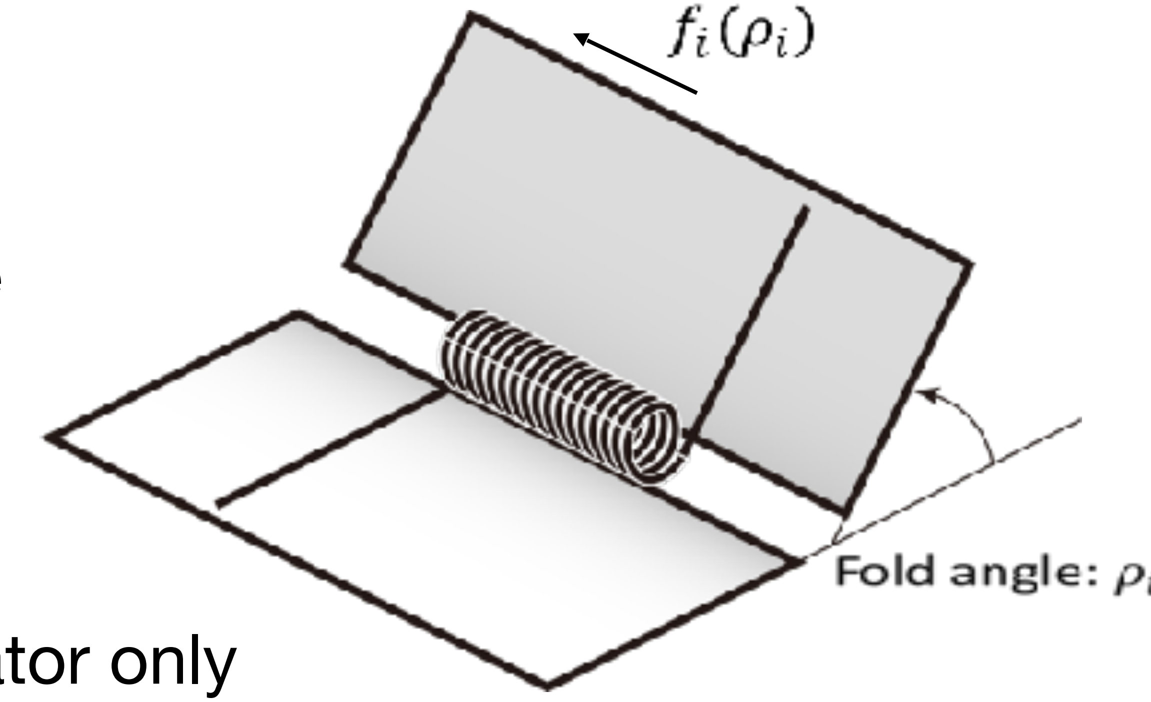
Note: (1) The flat state $\vec{\rho} = \vec{0}$ is a **singularity** (or a branch point) of the configuration space manifold. This means that folding from the flat state is harder than unfolding to the flat state. :-)

(2) The config space is symmetric about the origin (reversing MVs).

What is self-folding?

To self-fold, each crease needs an actuator of some kind.

Each actuator applies torque.
We call the collection of these torques at all the creases the **driving force** of the folding, which we can think of as a vector field on the configuration space.



In a **separable driving force**, each actuator only knows its own folding angle. I.e., $\vec{f} = (f_1(\rho_1), f_2(\rho_2), \ldots, f_n(\rho_n))$

A **conservative driving force** is the gradient of a potential function:

$$\vec{f} = -\nabla U(\vec{\rho})$$

For example, if we're lucky we could try the driving force:

$$f_i(\rho_i) = k_i(\rho_{\text{target}} - \rho_i)$$

What is self-folding?

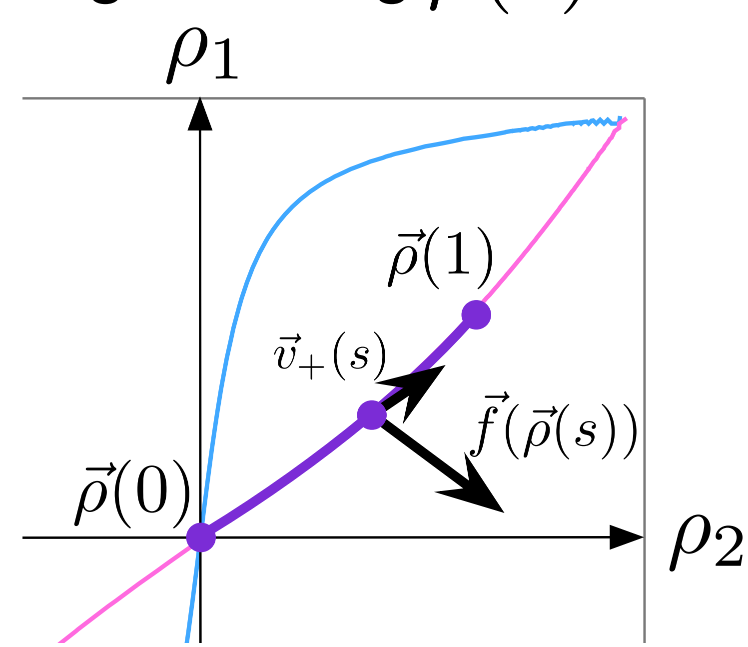
A **nice continuous rigid folding** $\vec{\rho}(s)$ from a rigid folding $\vec{\rho}(0)$ to $\vec{\rho}(1)$ is an arc-length parameterized piecewise C^1 curve in a configuration space. At each point there will be at most two tangent vectors:

$$\vec{v}_+(s_0) = \lim_{s \to s_0+} \frac{d\vec{p}(s)}{ds} \qquad \vec{v}_-(s_0) = \lim_{s \to s_0-} \frac{d\vec{p}(s)}{ds}$$

The set of all such tangent vectors for all nice continuous rigid foldings passing through a configuration point projected onto the unit sphere is called the **set of valid tangents**.

We define the **constrained forces** along a nice continuous rigid folding $\vec{\rho}(s)$ to be $f_+(s) = \vec{v}_+(s) \cdot \vec{f}(\vec{\rho}(s))$ (the forward force) and

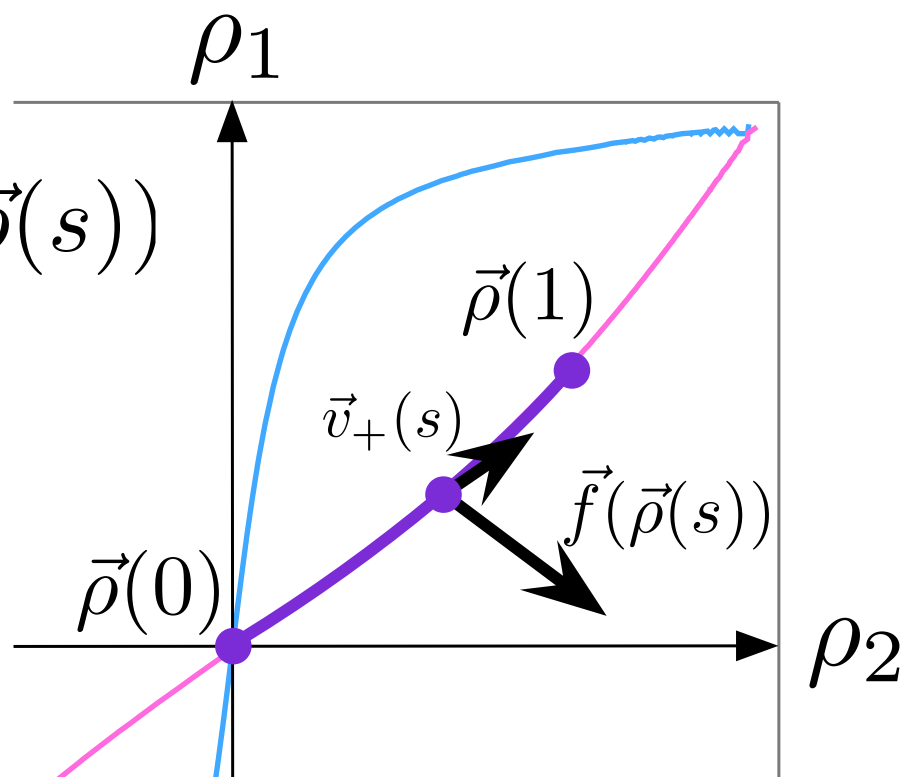
$f_-(s) = \vec{v}_-(s) \cdot \vec{f}(\vec{\rho}(s))$ (the backward force).



What is self-folding?

A nice continuous folding $\vec{\rho}(s)$ from a rigid folding $\vec{\rho}(0)$ to $\vec{\rho}(q)$ is **self-foldable** by driving force $\vec{f}(\vec{\rho})$ if for all $s \in [0, q)$ the forward force $f_+(s)$ at $\vec{\rho}(s)$ on the configuration space is positive and takes on a local maximum among the valid tangents at s.

That is, you want the dot product of $\vec{v}_+(s)$ and $\vec{f}(\vec{\rho}(s))$ to be positive. It makes sense!

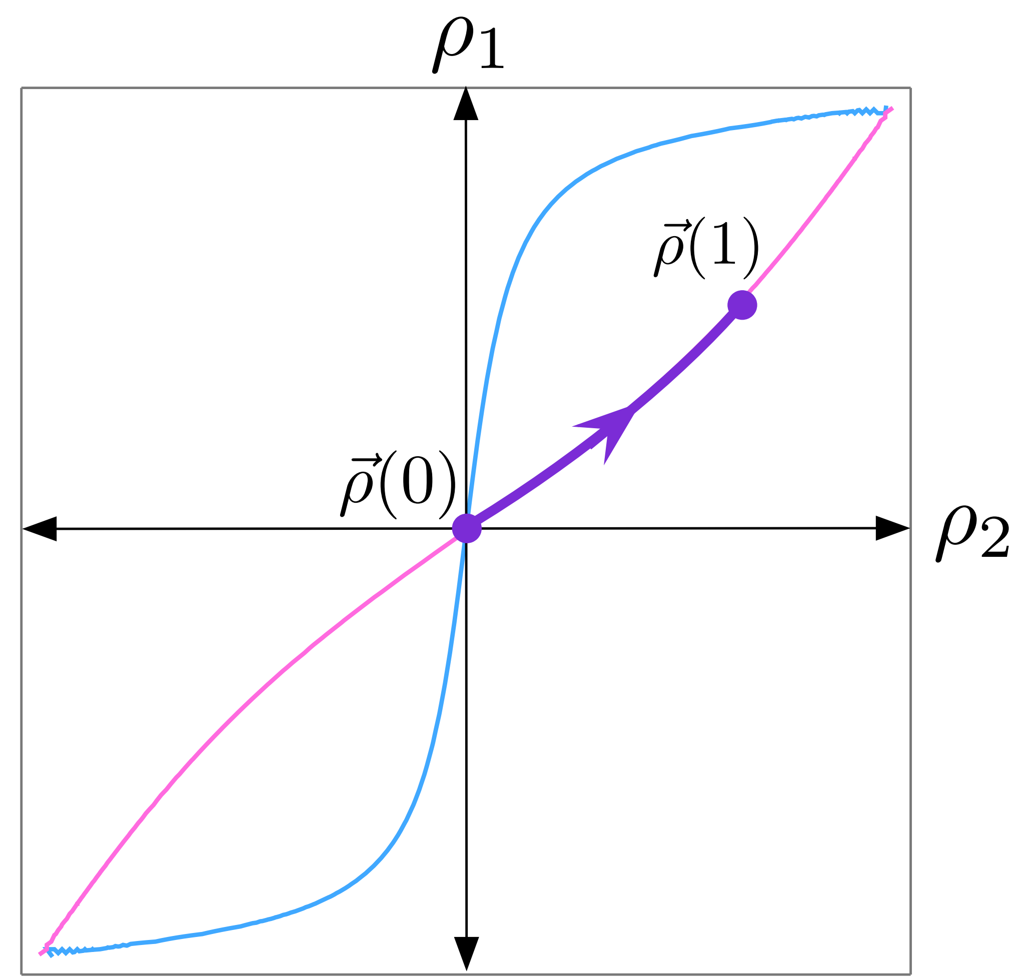


We say that a nice continuous folding $\vec{\rho}(s)$ is **uniquely self-foldable** if $\vec{\rho}(s)$ is the only nice continuous folding that is self-foldable by $\vec{f}$.

What is self-folding?

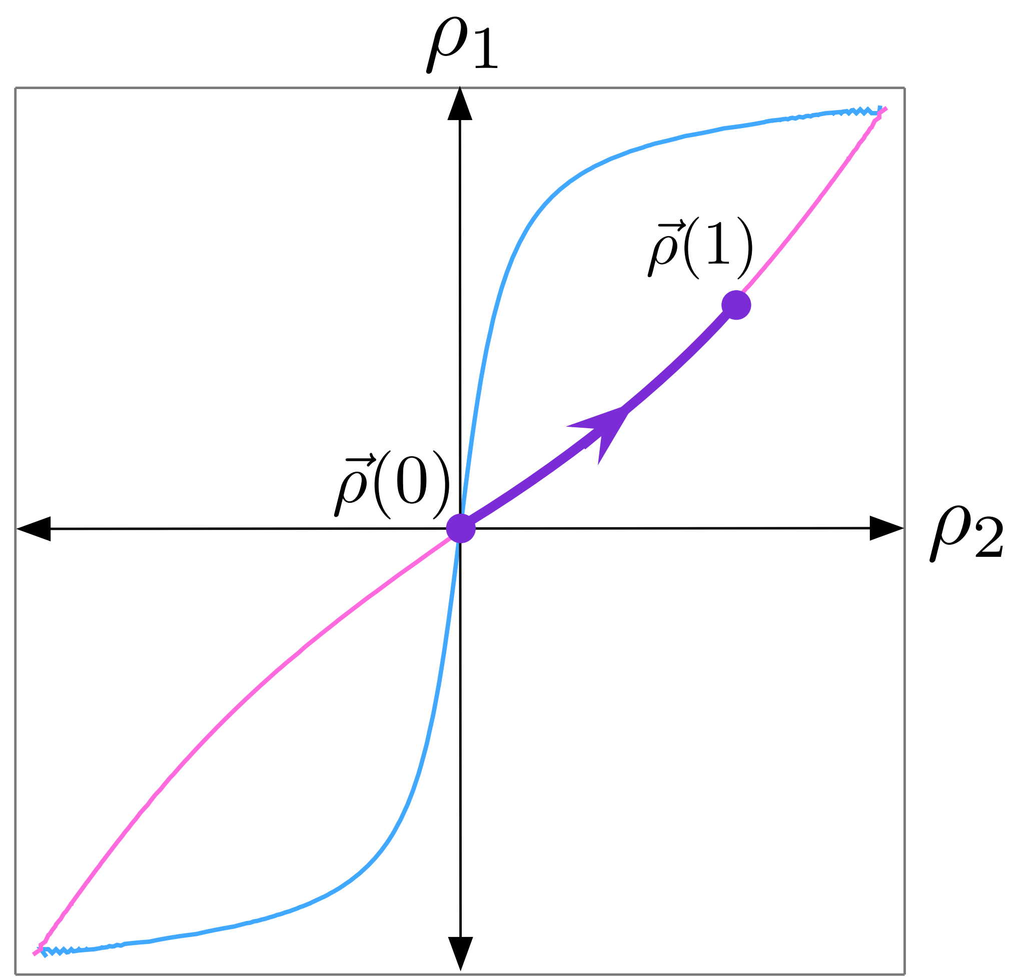
Theorem: For every rigidly, flat-foldable degree-4 vertex with arbitrary target and starting configurations, there exists a driving force that makes the vertex uniquely self-foldable.

Proof: Assume that the target $\vec{\rho}_T$ is on mode 1. We want to find a potential function $U(\vec{\rho})$ such that
(1) $U(\vec{\rho})$ monotonically decreases along mode 1 toward the target state and
(2) $U(\vec{\rho})$ monotonically decreases along mode 2 toward the flat state.



Back to the degree 4 case

Theorem: For every rigidly, flat-foldable degree-4 vertex with arbitrary target and starting configurations, there exists a driving force that makes the vertex uniquely self-foldable.



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The following works:

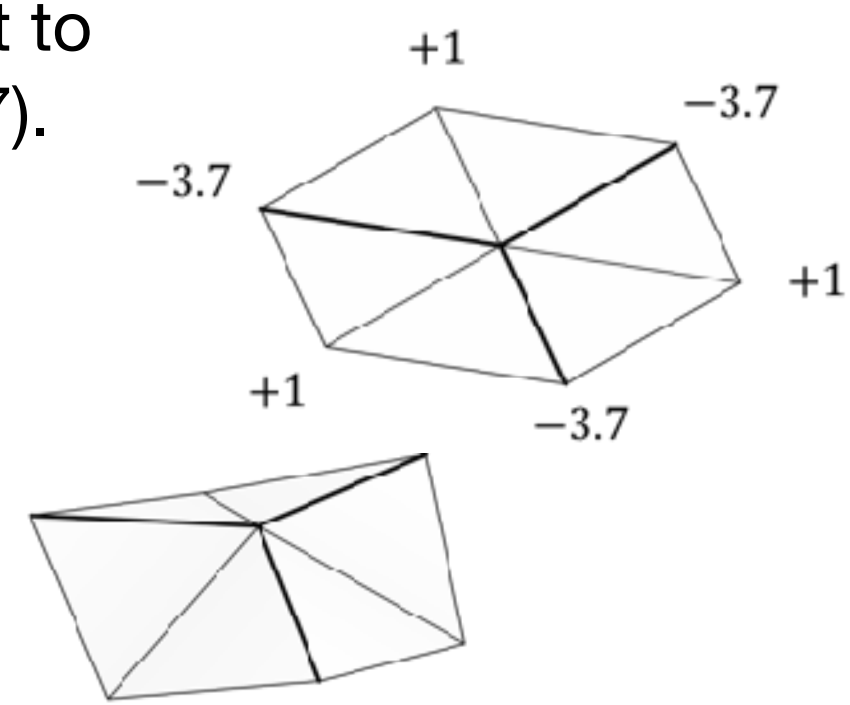
$$U(\vec{\rho}) = \frac{1}{2}\|\vec{\rho} - \vec{\rho}_T\|^2 = \sum_{i=1}^{4} \frac{1}{2}(\rho_i - \tau_i)^2$$

where τ_i are the coordinates of the target configuration.

Degree 6, symmetric vertex

Consider a symmetric, degree-6 vertex. We want to self-fold it to a target state (1, -3.7, 1, -3.7, 1, -3.7).

The "obvious" driving force would be to push the creases to be (V, M, V, M, V, M), right?



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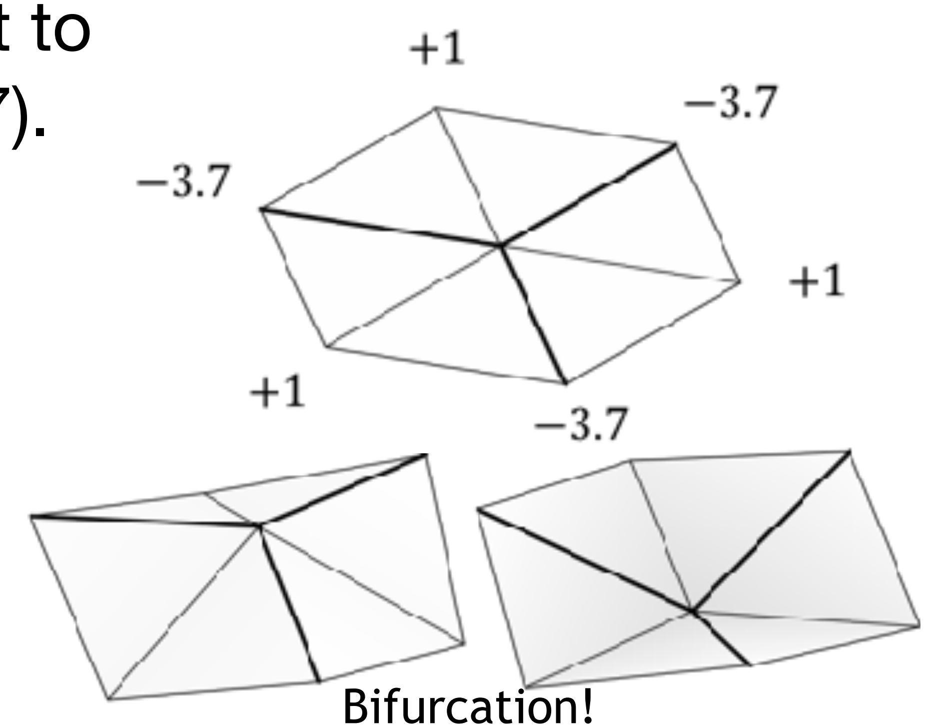
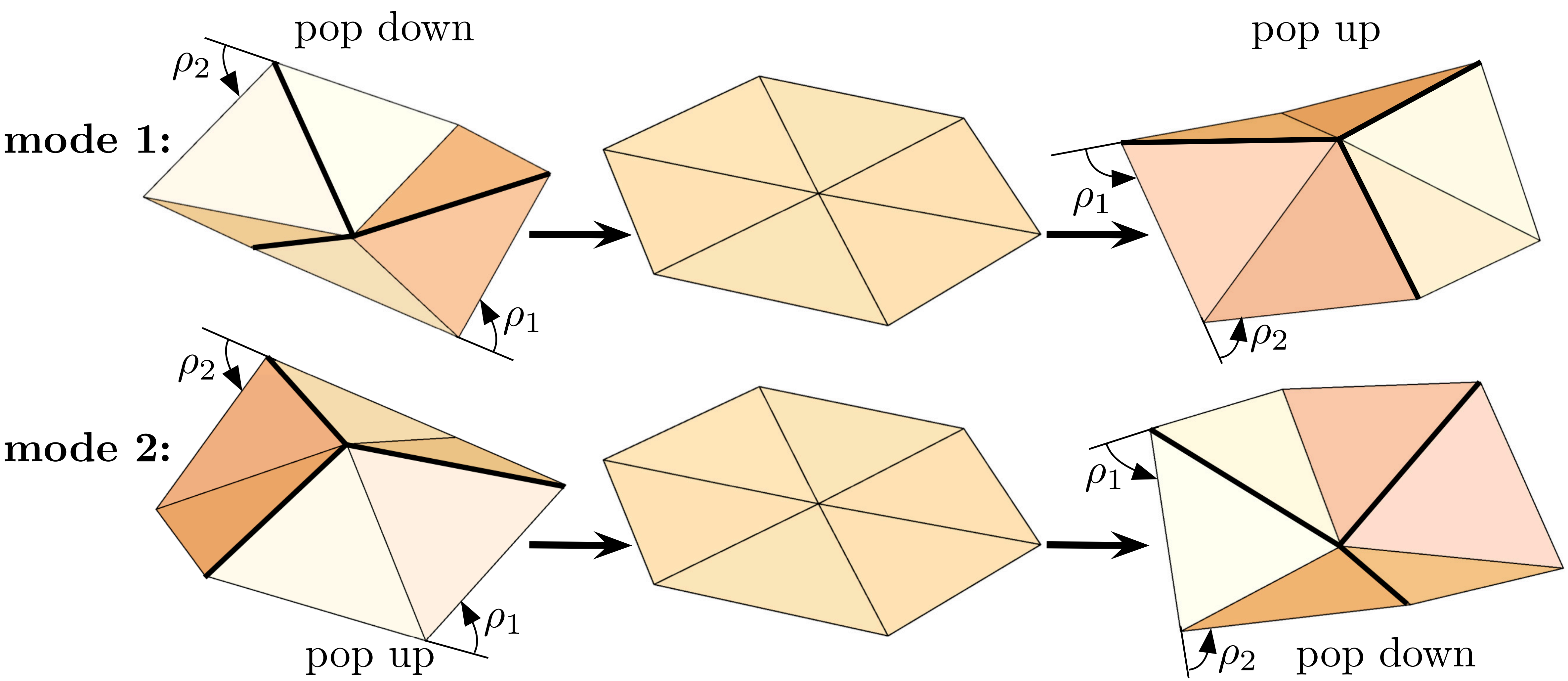


The "obvious" driving force would be to push the creases to be (V, M, V, M, V, M), right?

Wrong! The desired state is a pop-up vertex. But we can have a (V, M, V, M, V, M) state and get a pop-down vertex!

Degree 6, symmetric vertex

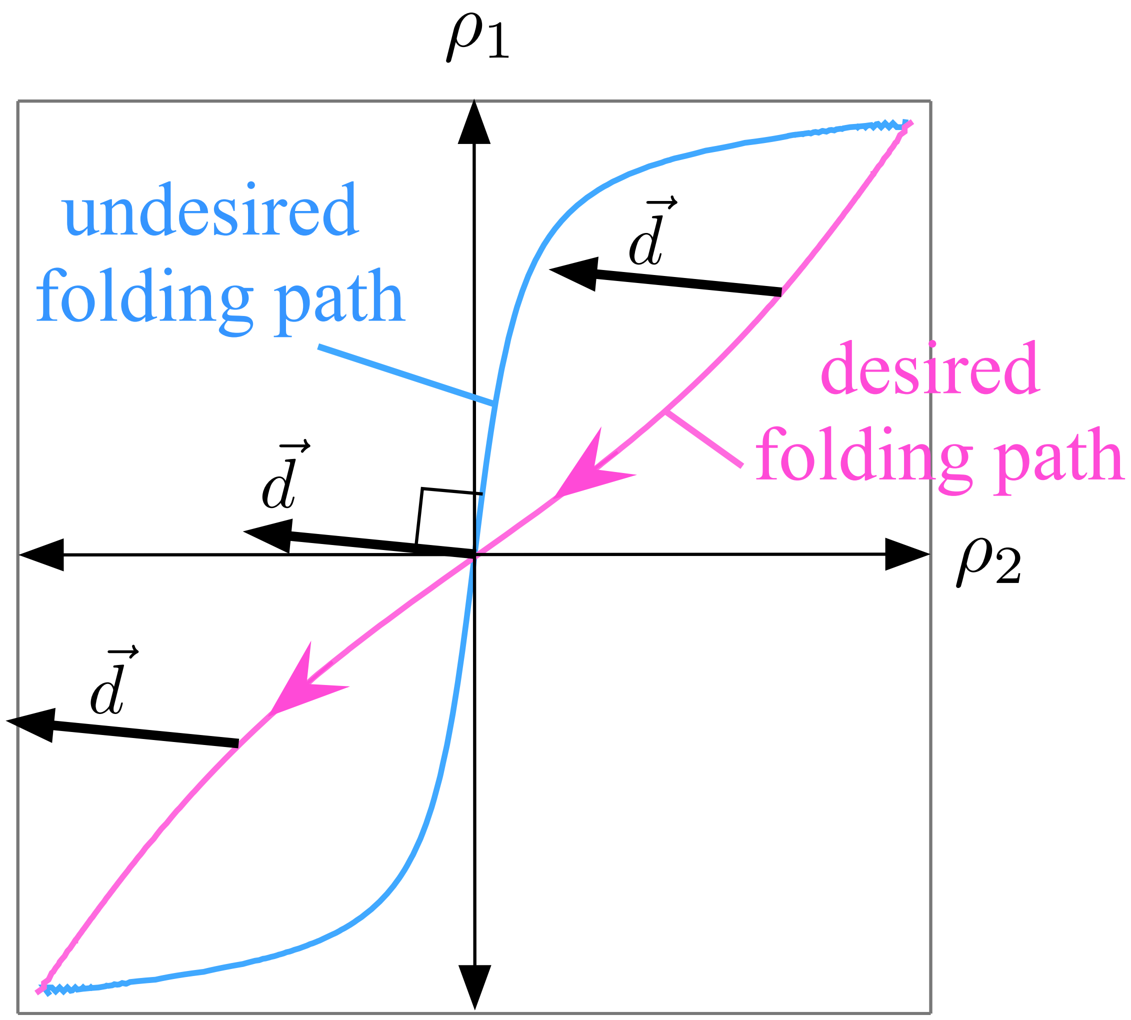
How do we “push” the creases to fold through singularities (like the flat, unfolded state) to the configuration we want?



Degree 6, symmetric vertex

How do we "push" the creases to fold through singularities (like the flat, unfolded state) to the configuration we want?

Here's the idea: Choose a driving force $\vec{f}$ that
(a) pushes the state along the desired path (making $f_+(s) > 0$), and
(b) is **orthogonal** to other, undesired manifolds at all singular points.



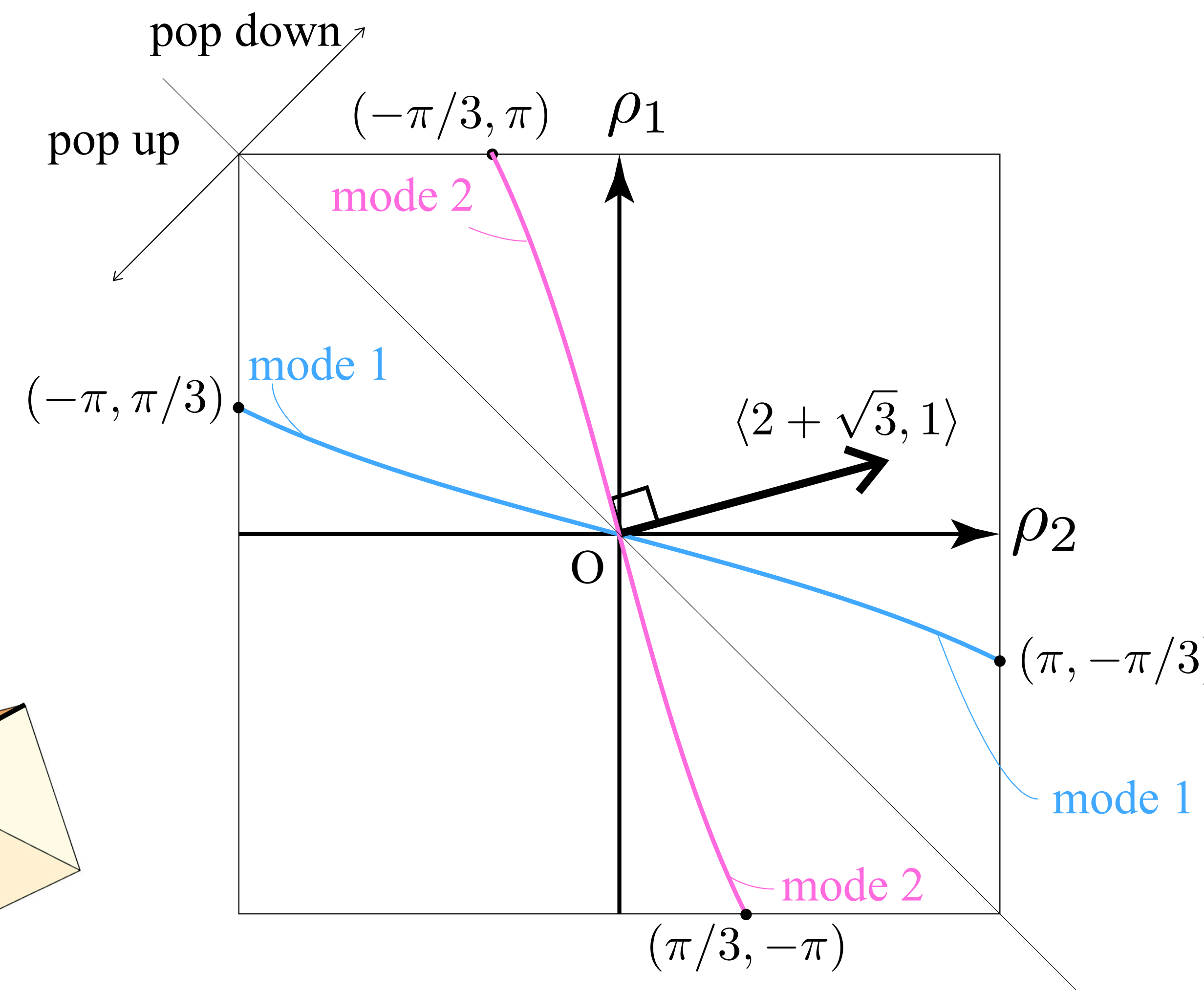
Degree 6, symmetric vertex

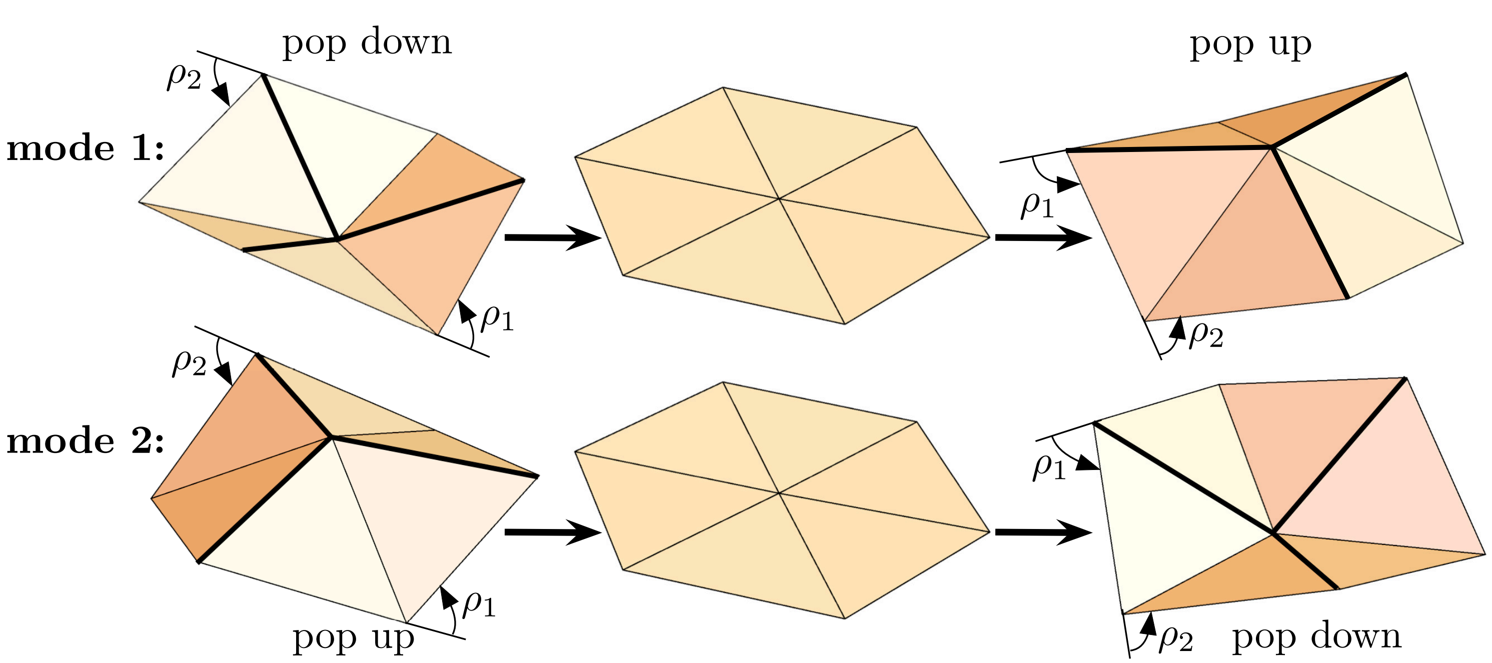
After some work, we can calculate the configuration space for the symmetrically-folding degree-6 vertex.

To self-fold along mode 1, for example, we need a driving force

$$\vec{f} = \langle 2+\sqrt{3}, 1\rangle$$

in order to be orthogonal to mode 2 at the flat state. (:o :o :o !!!!!)





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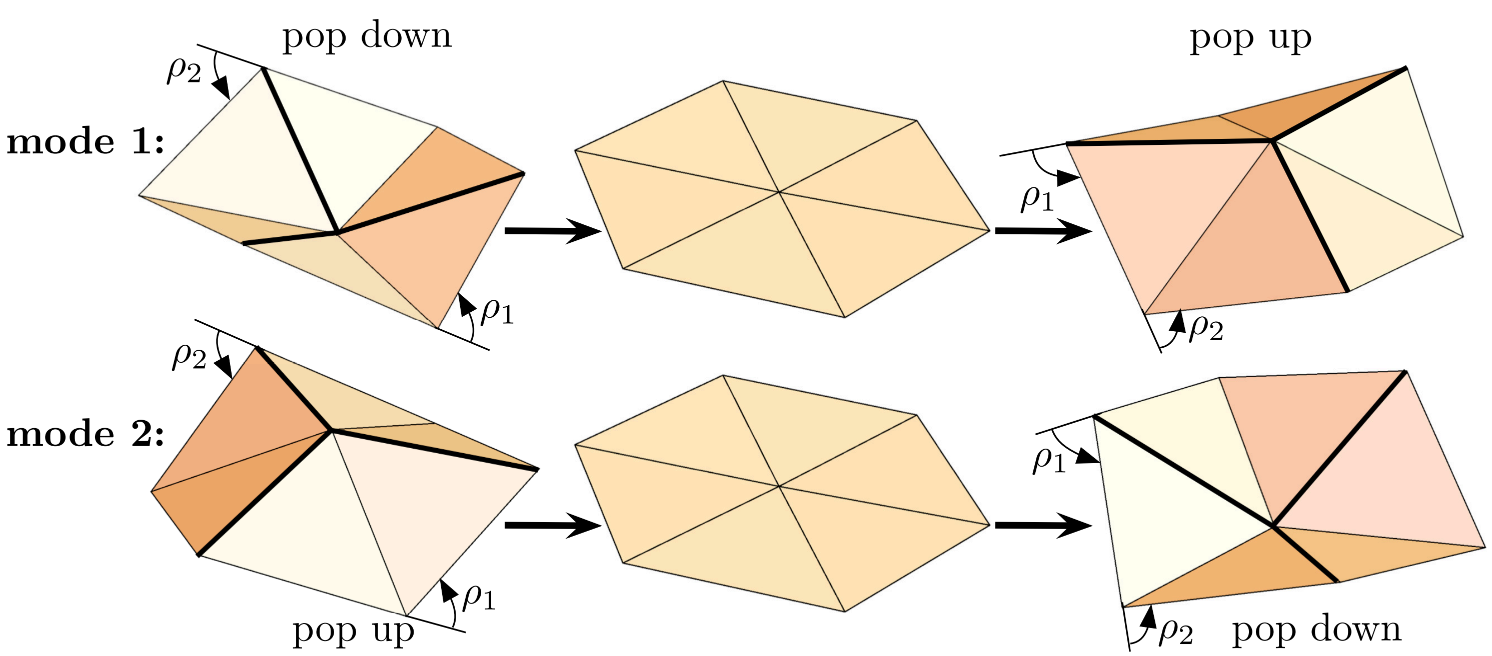
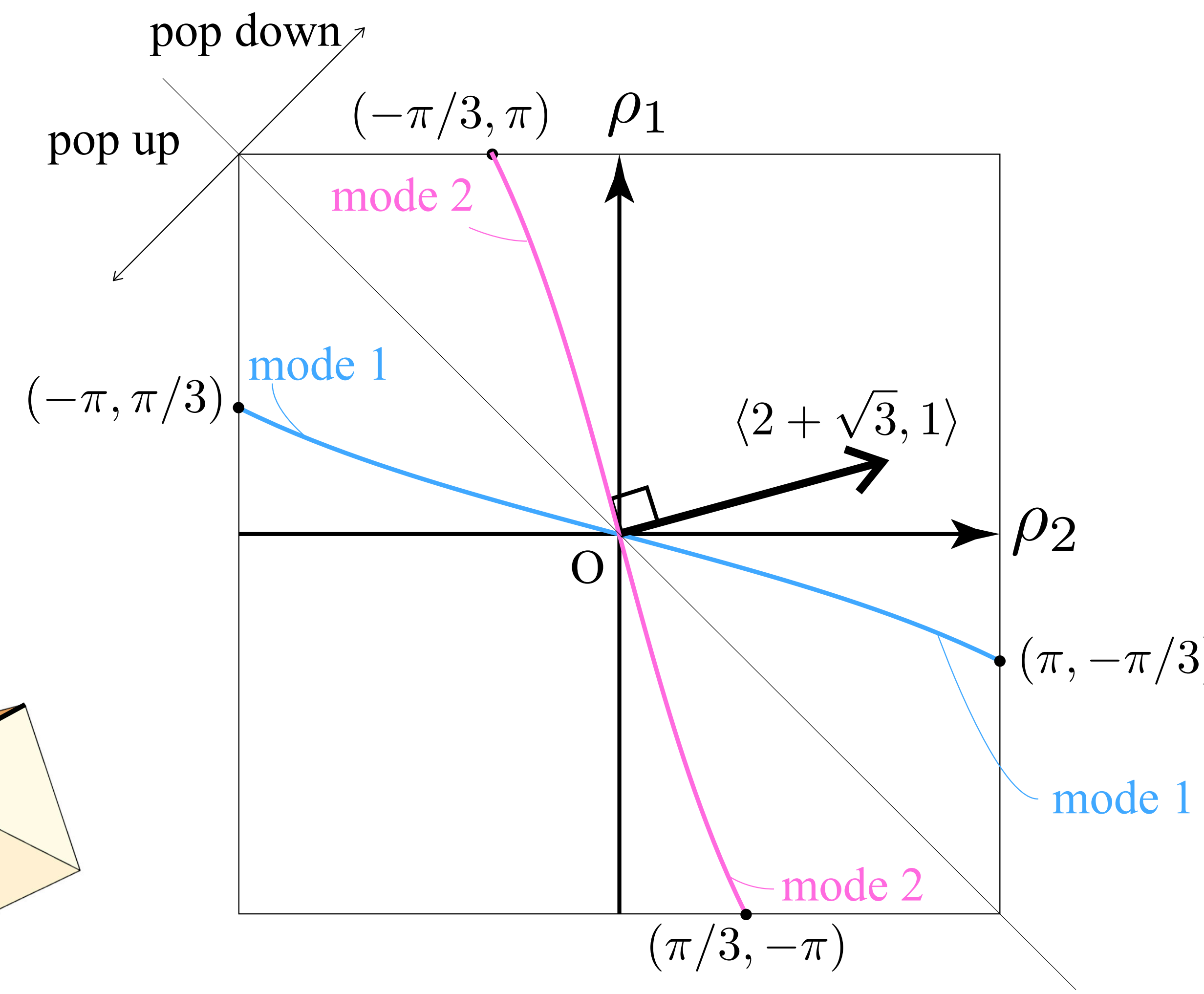
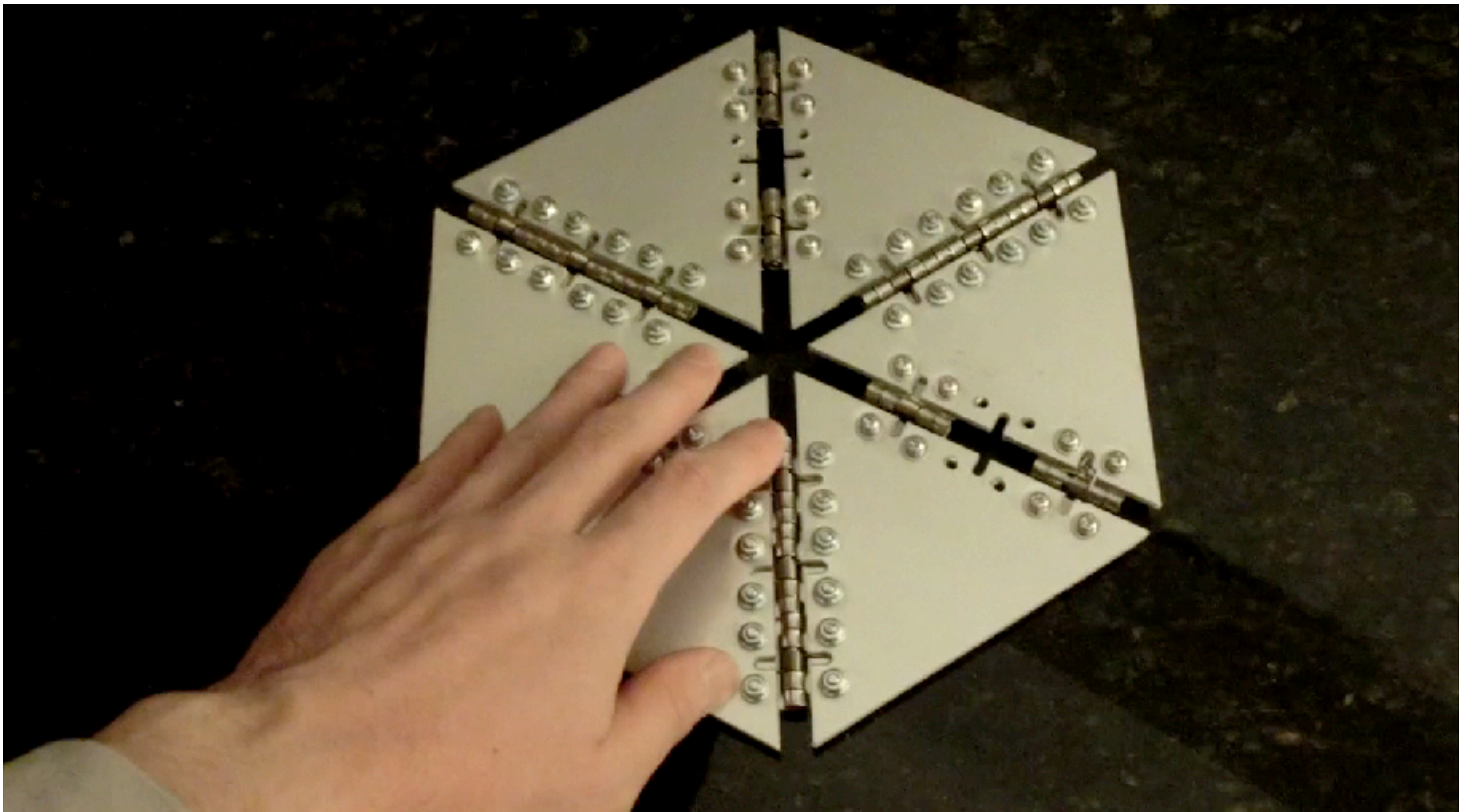




Thus we want to drive all creases to be **valleys**!!!

Degree 6, symmetric vertex

Amazingly, this actually works in experiments! Tomohiro made a physical model with loaded springs to model the two different valley torque strengths.



In conclusion

- We can also prove some multiple-vertex crease patterns can self-fold, while others cannot!

- See our paper: Self-foldability in rigid origami, *Journal of Mechanisms and Robotics*, Vol. 9, No. 2, 2017, 021008-021008-9, doi:10.1115/1.4035558

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