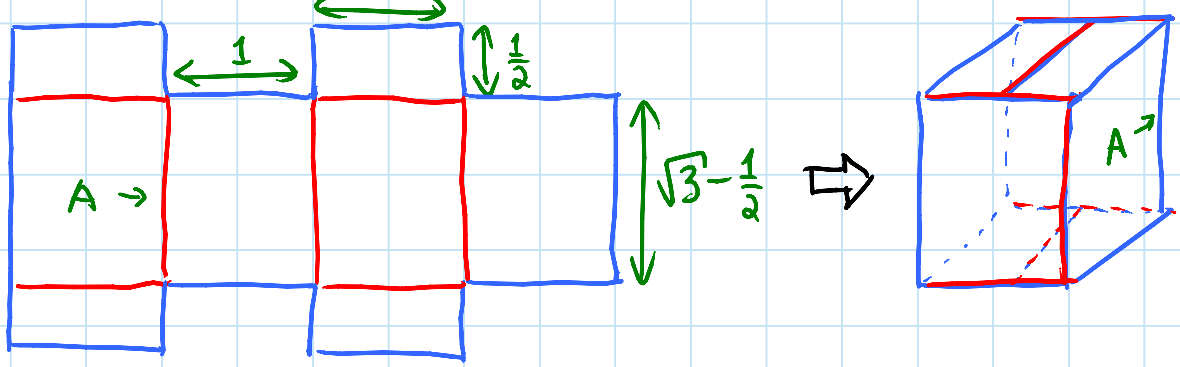
6.849	Lecture 17	Nov. 10, 2010

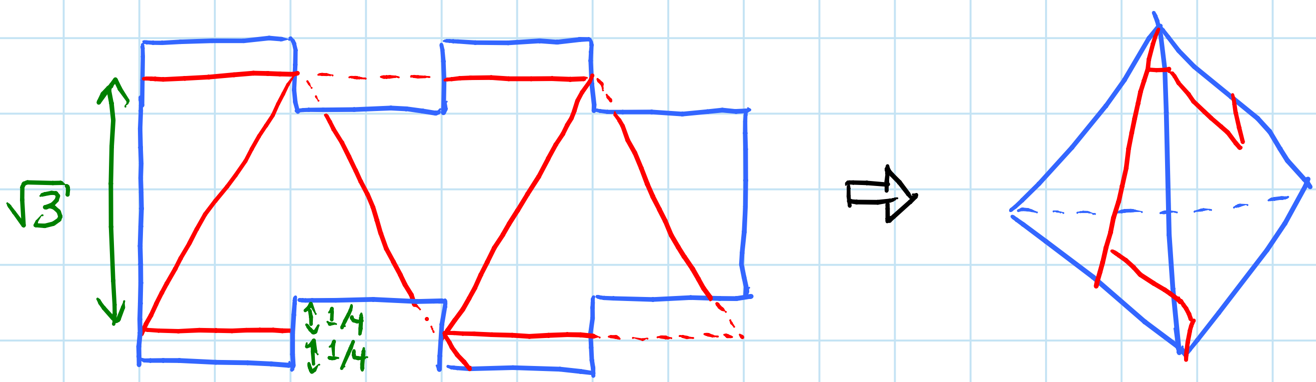
Polyhedron unfold/refold "dissection": (common unfoldings)

OPEN: which polyhedra P, Q are connected by
P → (general) unfolding → (general) gluing → Q?
or a sequence of unfolding, gluing, ...?
— true for any two Platonics? [M. Demaine, 1998]
e.g. regular tetrahedron & cube

Interesting examples:

— regular tetrahedron → $1 \times 1 \times \underbrace{1.232}_{\sqrt{3}-\frac{1}{2}}$ box possible
[Hirata 2000; GFALOP p. 425]

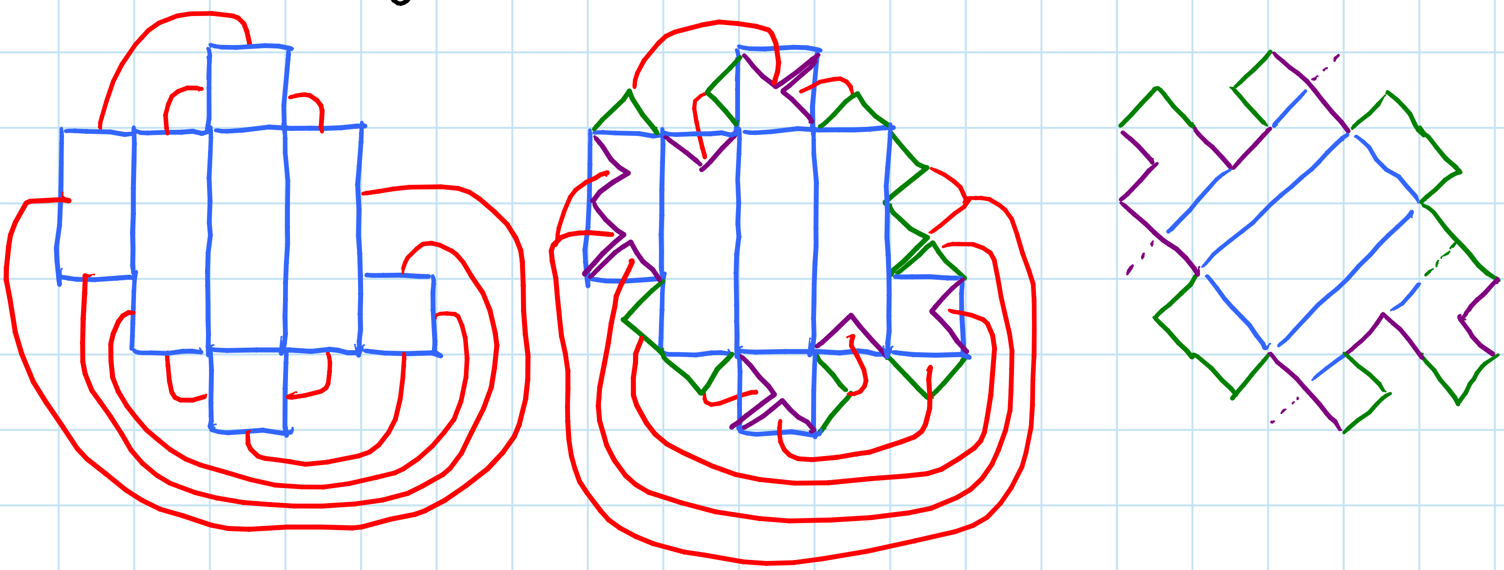




— regular (icosahedron, octahedron, cube) & tetrahedron with equal faces [Horiyama & Uehara 2010]
— Conjectured fractal common unfolding of regular tetrahedron & cube [Shirakawa 2010]

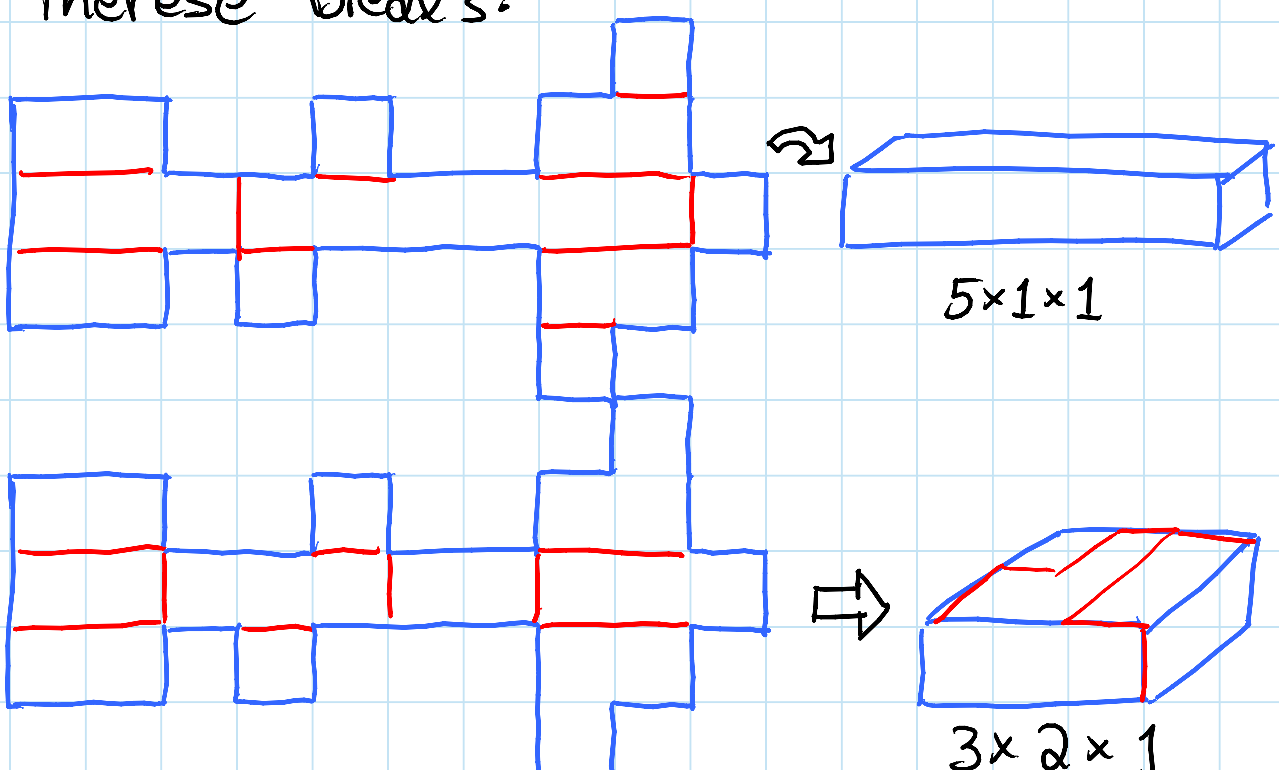
Box → box: [Biedl, Chan, Demaine, Demaine, Lubiw, Munro, Shallit 1999]

- Timothy Chan's solution: [GFALOP p. 425]



$\sqrt{2} \times \sqrt{2} \times 3\sqrt{2}$ → add tabs → $1 \times 2 \times 4$ box

- Therese Biedl's:



$5 \times 1 \times 1$

$3 \times 2 \times 1$

- also $8 \times 1 \times 1 \to 5 \times 2 \times 1$
- $\boxed{\text{OPEN}}$: when is $a \times b \times c \to d \times e \times f$ possible?

More boxes: [Uehara – CCCG 2008]

- random generation algorithm for orthogonal grid unfoldings (like Biedl's above)
  - found >25,000 so far!
- $1 \times 1 \times (6k+2) \Leftrightarrow 1 \times 5 \times 2k$ & $1 \times 1 \times (8k+11) \Leftrightarrow 1 \times 3 \times (4k+5)$ always possible
  $\Rightarrow$ infinitely many
- tiling example

$\boxed{\text{OPEN}}$: common unfolding of 3 boxes?

common surface area

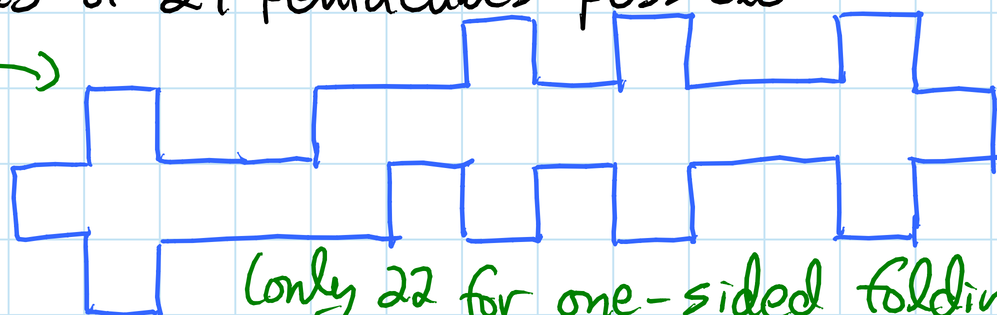
Cubigami: [Knuth & Miller 2005: Recent Toys]

common unfolding of all tree tetracubes

Generalization: [Aloupis, Benbernou, Bose, Collette, Demaine, Demaine, Douïeb, Dujmović, Iacono, Langerman, Morin 2010]

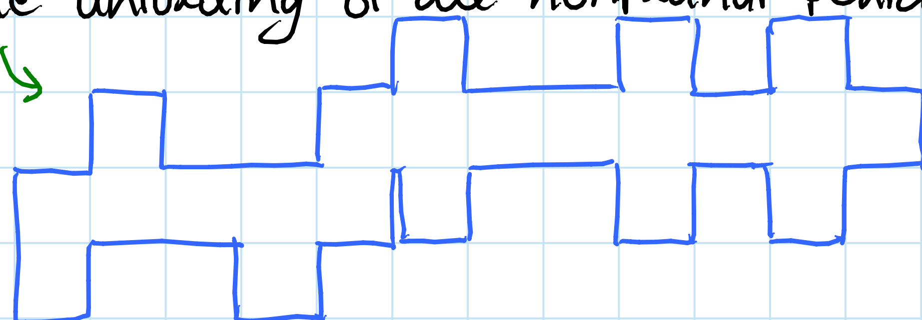
- no common unfolding of all tree pentacubes (trying all 1,099,511,627,776 unfoldings of )
- but 23 of 24 pentacubes possible

  e.g.:

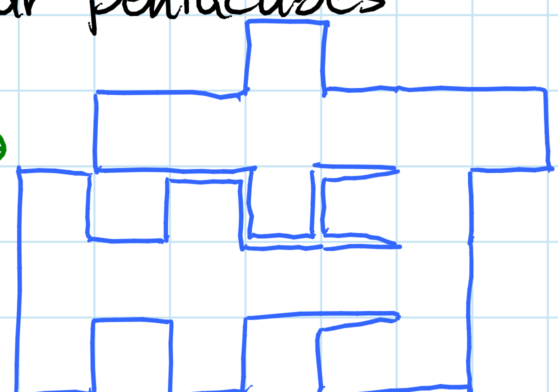


(only 22 for one-sided foldings)

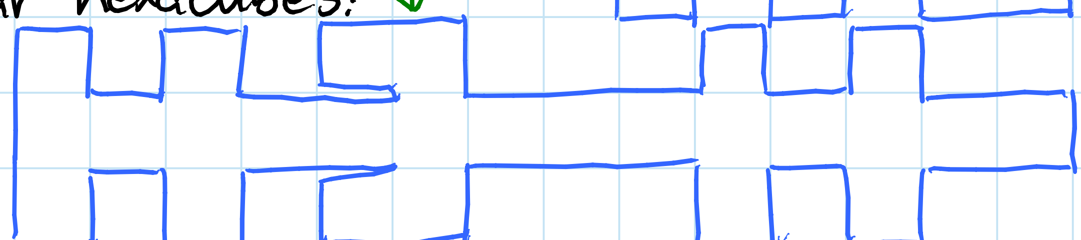
- unique unfolding of all nonplanar pentacubes (& 22 total)



- 492 unfoldings of all planar pentacubes (none fold to any nonplanar pentacube)



- no common unfolding of all path pentacubes
- planar hexacubes!



OPEN: 7?

- OPEN: any two polycubes without common unf.?

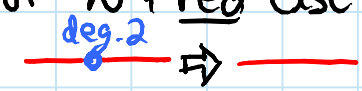
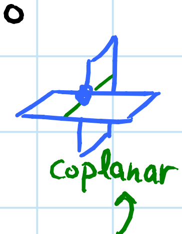
Orthogonal polygons vs. polyhedra:

can an orthogonal polygon with orthogonal creases fold into nonorthogonal polyhedra?

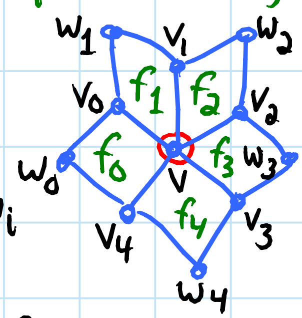
- answer depends on allowed genus

Genus	NonOrthog.?	Reference
$\emptyset$	NO	[Donoso & O'Rourke 2002]
1 & 2	NO	[Biedl et al. 2002]
3 − 5	OPEN	
6	YES	[Biedl et al. 2002]
≥ 7	YES	[Donoso & O'Rourke 2002]

Proof sketch that genus ≤ 2 ⇒ impossible:

- edge green if dihedral multiple of 90°; red else
- focus on red subgraph; coallesce 
- faces not planar, but face angles still mult. of 90° & dihedral angles still not mult. of 90°
- vertices have degree ≥ 4; deg. 4 special: 
- claim: $V \le 8(g-1) - \max\{D_{\neq 5}, F_{\geq 5}/2\}$

  (proof uses Euler's Theorem + analysis of special vxs.)



- degree ≥ 4 & V ≥ 1 ⇒ V ≥ 5
- claim ⇒ $D_{\neq 5} \le 3$ ⇒ $D_5 \ge 2 \to v$
- $V \ge 6 \Rightarrow F_{\geq 5} \le 4 \Rightarrow$ some f_i degree $4 \to w_i$
- $V \ge 7 \Rightarrow F_{\geq 5} \le 2 \Rightarrow \cdots \Rightarrow$ all f_i degree 4
- $w_i \neq w_{i+1} \Rightarrow \geq 3$ distinct $\Rightarrow V \ge 9$ but $V \le 8$ □

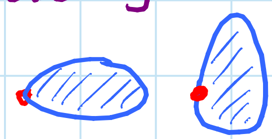
Smooth Alexandrov: [Pogorelov 1973]

every convex metric, topologically a sphere, is realized by a unique convex surface, possibly degenerating to flat doubly covered convex shape

- proof idea: take limits of polyhedral approximation

D-forms: [Tony Wills; Pottmann & Wallner 2001]

- take two convex smooth shapes of the same length



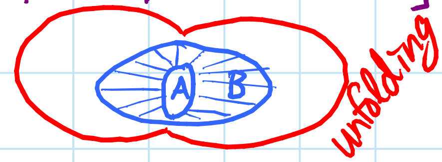
- identify two boundary points of two shapes
- glue around from there
- smooth Alexandrov ⇒ get convex surface

Results: [Demaine & Price 2009]

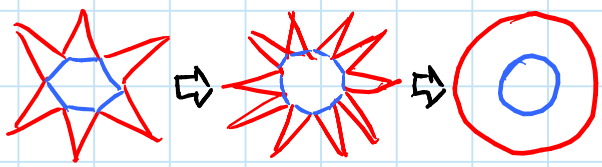
- no creases except along seam
- form = convex hull of its seam

Unfolding smooth prismatoids: [Benbernou, Cahn, O'Rourke 2004]

= convex hull of two parallel smooth convex shapes



- keep bottom B, $\underbrace{\text{unroll side ribs}}_{\text{volcano unfolding}}$, $\underbrace{\text{place top A}}_{\text{nontrivial}}$
- simple example: pyramid (A = single point)



limit

- surface area not preserved (unlike regular unfolding)
- like source unfolding of sphere (below)
- always works, yet:

  $\boxed{\text{OPEN}}$: does every (discrete) prismatoid have an edge unfolding?

  from Lecture 13

Wrapping smooth surfaces with flat paper:

[Demaine, Demaine, Iacono, Langerman 2007]

- impossible with finite number of creases to make all points have nonzero curvature
- idea: allow folding to shrink some intrinsic distances on paper (contractive mapping)
  - simulate by crinkling paper:

Burago & Zalgaller Theorem: [1996]

any contractive C^2-immersion of a polygon (or more generally, a polyhedral metric) admits a $C^\emptyset$-approximation (within ε $\forall \varepsilon > 0$) by isometric piecewise-linear $C^\emptyset$-immersions

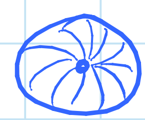
- also, noncrossingness is preserved

Stretched path = isometrically folded (unshrunk) path

- optimal wrapping should have one; else scale

Stretched wrapping = stretched path between every two points

Source wrapping of convex surfaces:



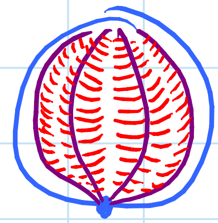
- stretched paths along all shortest paths from x
- cut at ridge tree / cut locus
- e.g. unit sphere $\Rightarrow$ disk of radius π, area π^3, perimeter $2\pi^2$

Strip wrapping:

- e.g. unit sphere $\Rightarrow$ area $\to 4\pi$, perimeter $\to \infty$

Wrapping smooth surfaces with flat paper: (cont'd)

Petal wrapping of sphere:

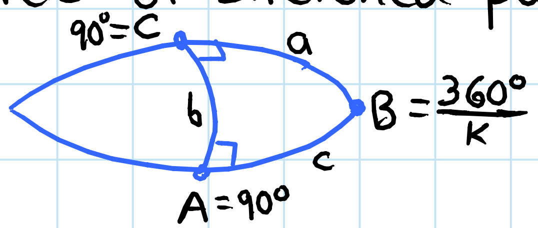


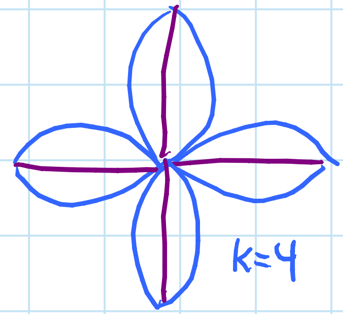
Voronoi diagram

- k stretched paths from south to north pole
- perpendicular stretched paths till meet another

$\Rightarrow$ $360°/k$ "orange peel" for each primary path

- depth-2 tree of stretched paths





- Spherical Law of Cosines $\Rightarrow$

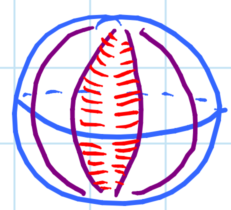
$$\cos C = -\underbrace{\cos A}_{0} \cos B + \underbrace{\sin A}_{1} \sin B \cos c$$

$$\Rightarrow \sin C = \sqrt{1 - \sin^2 B \cos^2 c}$$

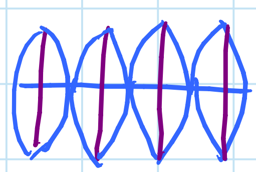
- Spherical Law of Sines $\Rightarrow$ $\dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$

$$\Rightarrow b = b(c) = \arcsin \frac{\sin c}{\sqrt{\frac{1}{\sin^2 B} - \cos^2 c}}$$

Comb wrapping: depth-3 tree



- stretched path around equator
- k paths from equator to north pole, ditto for south pole
- perpendicular stretched paths
- same petals, different gluing



Wrapping smooth surfaces with flat paper: (cont'd)

Both petal & comb wrappings: area $\to 4\pi$
perimeter $\to \infty$

Real Mozartkugel wrappings:

- square containing 4-petal [Fürst]
  - $\sqrt{2}\pi \times \sqrt{2}\pi$
  - $\Rightarrow$ area $2\pi^2$, perimeter $8\pi/\sqrt{2} \approx 5.7\pi$
- rectangle containing 4-comb [Mirabell]
  - $\pi \times 2\pi$
  - $\Rightarrow$ area $2\pi^2$ (!), perimeter 6π >

Better Mozartkugel wrappings:

- equilaterial triangle containing 3-petal has area $1.9983\pi^2$ ($\approx 0.1\%$ improvement)

- packing 3-petals $\Rightarrow$ $1.6033\pi^2$ area each
- packing k-combs $\Rightarrow$ $1.3333\pi^2$ area each
- vs. optimal: $4\pi = 1.2732\pi^2$

OPEN: what is the best area for given perimeter? (Pareto curve)

OPEN: what is the minimum possible perimeter?

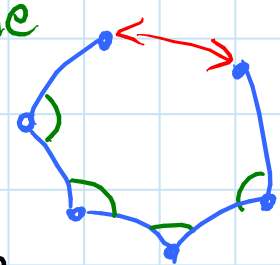
OPEN: what about smooth surfaces other than the sphere?

How to prove contractiveness of these wrappings?

Cauchy's Arm Lemma on a growing sphere:

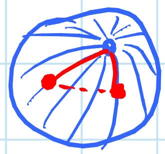
[Abel, Charlton, Collette, Demaine, Demaine, Langerman, O'Rourke, Pinciu, Toussaint 2008]

e.g. plane



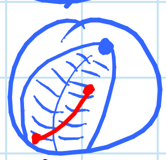
open convex chain on a sphere, redrawn on a larger-radius sphere with matching edge lengths & angles, increases the length of the closing edge

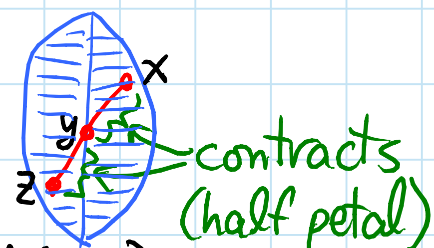
Contractive corollaries:



- source wrapping

- half petal





- petal
  - plane: $d(x,z) = d(x,y) + d(y,z)$
  - sphere: $\geq d'(x,y) + d'(y,z)$
    $\geq d'(x,z)$ (Δ inequality)
- petal unfolding: same argument (shortest path between x & z)