Locked & unlocked chains of planar shapes:
rigid objects in place of bars

- simple locked examples:
  [M. Demaine 1998]

3 triangles

Adorned chain: view shapes as adornments attached to bar connecting hinges
- underlying chain linkage
- some flexibility in first & last shape

Adornment = shape + base
- shape = simply connected compact planar region
- base = line segment connecting 2 boundary pts.
  require base to be contained in shape
Slender adornment = walking along boundary from one base endpoint to the other monotonically increases distance to former (and decreases distance to latter)

= all inward normals hit the base (for piecewise-differentiable shapes)
= (possibly infinite) union of half-lenses: intersection of disks centered at base endpoints and halfplane on one side of base
  (⇒ can define slender hull = union of half-lenses thru every point in adornment)

Slender ⇒ not locked: expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Intuition: consider two touching adornments
- draw collinear inward normals from touching point x
- resulting points a & b expand (vertices expand ⇒ points on bars expand)
⇒ two copies of x locally expand
- in reality, this argument is tricky; can stay equal, to first order
- possible with strict expansiveness [see SoCG 2006 proof]
Symmetric case: adornments reflectionally symmetric about their bases
⇒ slender adornment = union of lenses

Stronger result for this case:
  instantaneous expansive motion of underlying chain linkage preserves nonintersection of slender adornments
Proof: take any two lenses of different adornments
  - nonintersecting before the motion
  i.e. four disks have empty intersection

Kirszbraun's Theorem: [1934]
  if we instantaneously translate $n$ disks with an empty $n$-way intersection according to an expansive motion on their centers,
  then they still have empty intersection

(annoying detail: Kirszbraun's disks include their boundary, but our disks might kiss — but Kirszbraun's Theorem holds for disks excluding their boundaries, by taking limits of slightly smaller disks)

If initially intersecting, can show area of union only increases [Bezdek & Connelly 2002 ~Kneser-Poulsen conj.]
Proof that slender \( \Rightarrow \) not locked: (general case)

- not true for instantaneous: \( \includegraphics[width=0.5in]{triangle.png} \Rightarrow \includegraphics[width=0.5in]{triangle.png} \)

- take any 2 half-lenses of different adornments
- consider transition time from not intersecting to intersecting \( \Rightarrow \) touching

- 3 types of touching:
  1. bases of both
     - non-intersection guaranteed by underlying chain linkage
  2. base of one
     - can add symmetric lens of other, & just consider base of first (\( \times \))
     - no intersection by symmetric case
  3. base of neither
     - can add symmetric lens of both
     - again no intersection by symmetric case \( \square \)

Carpenter's rule theorem \( \Rightarrow \) straighten/convexify any slender-adorned (non-self-touching) chain
\( \Rightarrow \) connected config. space of open chains
- not true of closed chains:
**OPEN:** which adornments never lock in a chain?
(like slender)

**Triangles:** not locked if angles opposite base $\geq 90^\circ$
(right or obtuse)

- locked (nearly) identical equilateral triangles:

  + can stretch/shrink in y coord. to make locked example with any angle $< 90^\circ$

**Proof:** show self-touching version rigid
$\Rightarrow$ strongly locked  [Lecture 11]

**Rule 1**

**Rule 2**

**Lemma:** any motion of a convex polygon decreases $\geq$ two angle $\Rightarrow$ RIGID
Locked triangles proof: (cont'd)

- Clearly rigid if zero-length struts were bars
- Set $s(AB) = -s(AB') < 0 \Rightarrow A$ in equilibrium
- Set $s(BC) = s(AB) = -s(B'C') = -s(AB') < 0$
  $\Rightarrow$ force on $B, B'$ vertical $\Rightarrow$ in equilibrium if
  set $s(B, AB') = s(B, B'C') < 0$ appropriately
- Set $s(C'D'), s(D'DC), s(D'DE) < 0$ unique up to scale
  to put $D'$ in equilibrium; scale very small
- Set $s(CD) = -s(C'D') \Rightarrow D$ in equilibrium (inverse of $D'$)
- $s(BC) < 0$ dominates $s(CD) \Rightarrow$ can set $s(C, C'B') &$
  $s(C, C'D') < 0$ to put $C$ & hence $C'$ in equilibrium $\square$

OPEN: locked chain of unit squares?
Hinged dissections: [Abbott, Abel, Charlton, Demaine, Demaine, Kominers 2008/2010]

There’s an open chain of hinged polygons that folds continuously into any desired finite set of polygons of equal area (without collision).

Idea:
1. Start with any dissection — no hinges set of polygons that can be assembled into each target polygon.
2. Hinge arbitrarily (or to match one target).
3. Subdivide pieces & add hinges to enable each desired assembly.
4. Subdivide to make pieces slender ⇒ motion.

1: [Lowry 1814; Wallace 1831; Bolyai 1833; Gerwijn 1833]
   - Cut each polygon into triangles
   - Cut each triangle into rectangle.
   - Dissect each rectangle into rectangle of height \( \varepsilon \) [Montucla 1778]
     (this is the hard step ~ skipped here)
   - String rectangles from one polygon into one long height-\( \varepsilon \) rectangle
   - Overlay these cut patterns of \( 4/3 \times 3 \) rect.
3. maintain tree hinging
   - key step: effectively move rooted subtree to attach at any other vertex

   - 2 of these ops. brings two vertices together
   - repeat until vertices together for target
   - repeat for each target polygon

# pieces: without care, can roughly double for each step
   - can be improved with care

4. triangulate pieces
   - cut each Δ at in-center
     - obtuse
   - connect into chain by "walking along the outside of the tree" (Euler tour)
     - slender chain
     - not locked