Locked linkages: recall

- chains
- trees
- 2D never locked ✓
- can lock ✓
- 3D can lock
- 4D+ never locked
- TODAY

Algorithms for unfolding 2D chains

① ordinary differential equation given by (canonical) expansive infinitesimal motion

\[ \frac{dE}{dt} = d \]

[Connelly, Demaine, Rote 2000, 2002]

- strictly expansive (other than)
- one step in poly. time: convex program
- many steps: inaccurate (without projection)
- OPEN: how many? pseudopolynomial?

② pointed pseudotriangulations [Streinu 2000, 2005]

- expansive \( \Rightarrow \) maximal edge set on given points with \( > 180° \) angle at every vertex
- \( n^{O(1)} \) steps
- one step follows 1D.O.F. linkage \( \Rightarrow \) delete edge of convex hull
- best algorithm is exponential
- OPEN: are pseudotriangulations easier than general 2D linkages?
  (e.g. they are noncrossing)
- PROJECT: implement this algorithm
Algorithms for unfolding 2D chains: (cont’d)

3 energy

- not expansive
- one step is $O(n^2)$ & exact on real RAM
- pseudopolynomial number of steps
  \[ \text{poly. in } n \& r = \text{max. dist.}/\text{min. distance} \]

Approach:

- define energy function on configurations:
  \[ E(C) = \sum_{\text{edge } vw} \sum_{\text{vertex } u} \frac{1}{d(u,vw)} \]

- any energy-decreasing motion avoids crossings: approaching $O(\text{dist})$ shoots $E \rightarrow \infty$
- expansive motion decreases energy
  (in fact, every term)

⇒ energy-decreasing motions exist

⇒ downhill gradient of energy exists: $-\nabla E$
- computable in $O(n^3)$ time
- lower bound gradient, upper bound curvature

⇒ $\Theta(n^{123} r^{-41})$ step bound (!)

OPEN: improve step bound (likely not hard)

OPEN: $n^{O(1)}$ step bound possible? conjecture no

OPEN: is minimum-energy configuration unique?

for equilateral polygons, it’s a regular n-gon
Single-vertex rigid origami: [Streinu & Whiteley 2001] every folded state of a single-vertex crease pattern can be folded rigidly (continuously, faces staying rigid)

\[ \bigcirc \bigcirc = \bigcirc \bigcirc \] linkage folding!

Spherical Carpenter's Rule Theorem: [Streinu & Whiteley]
closed chain of total length \(2\pi\) on unit sphere has a connected configuration space
- proof based on projective invariance of infinitesimal rigidity
- \( \text{length} \leq 2\pi \) \( \Rightarrow \) lie in hemisphere
  \( \Rightarrow \) can project to plane
- \( \text{length} = 2\pi \) \( \Rightarrow \) convex config. = equator
- \( \text{length} < 2\pi \) \( \Rightarrow \) 2 convex configs. (cw & ccw)
  \( \Rightarrow \) 2 connected components of configs.
- \( \text{length} > 2\pi \) \( \Rightarrow \) no convex configuration

Touching case (e.g. flat folding) handled by recent self-touching Carpenter's Rule Theorem [Abbott, Demaine, Gassend 2007]
Locked 2D trees:

- deg. $\geq 5$
- diameter 4
- only 1 degree-3 vertex

[Biedl et al. 1998/]  [Poon 2005]  [Demaine, Rote 2002]

[V] 8 edges
- linear

[Ballinger, Charlton, Demaine, Demaine, Iacono, Liu, Poon 2009]

- locked linear trees have
  - $\geq 8$ edges
  - $\geq 9$ edges or diameter $\geq 6$

[Ballinger et al.]

OPEN: 8 edges minimal for nonlinear? 14 edges minimal for orthogonal?
OPEN: characterize locked linkages
e.g. locked trees in 2D or chains in 3D
  - polynomially solvable?
  - special case: linear trees

Related problem: can you fold config. A \rightarrow config. B?
  - PSPACE-complete for 2D trees & 3D chains
    [Alt., Knauer, Rote, Whitesides 2004]
  - but their reductions use locked linkages as gadgets — so all locked
Infinitesimally locked linkages \cite{Connelly, Demaine, Rote 2002}

Intuition: in many locked examples (particularly 2D), as gaps get smaller, so do valid motions

Locked within $\varepsilon$ = configuration from which it is impossible to get farther than $\varepsilon$ in configuration space

Rigid = locked within $\emptyset$ — but trees are never rigid... right?

Self-touching configuration allows infinitesimal gaps: geometric overlap, distinguished combinatorially

- now can be rigid

Return to nontouching: rigidity $\Rightarrow$ "strongly locked"

Strongly locked = sufficiently small perturbations are locked within $\varepsilon$, for any $\varepsilon > 0$

$S$-perturbation = move vertices within $S$-disks, preserving combinatorial sidedness

Every self-touching has a (non-self-touching) $S$-perturbation, for all $S > 0$ [Ribó Mor, PhD 2006]
Proof based on “sloppy rigidity”: [Connelly 1982]
if relax the edges in a rigid tensegrity (bars can change length by $\delta$, struts can shrink by $\delta$, etc.)
then still can’t move more than $\varepsilon$

Infinitesimally locked linkages: (cont’d)

Infinitesimal rigidity:
- implies rigidity
- “zero-length strut” (linear inequality):
  $u$ should remain right of $vw$
- sometimes nonconvex:

$\Rightarrow$ conservative polynomial test (drop constraints) or exponential test (split into 2 convex)
- analogs of equilibrium stress & duality
- even Maxwell-Cremona [Ribó Mor, PhD 2006]
- nice proofs by hand: positive stress on struts
  $+ \text{ underlying linkage rigid}$
  $\Rightarrow \text{ inf. rigid}$
  $(\Rightarrow \text{ rigid})$
  $(\Rightarrow \text{ strongly locked})$

PROJECT: implement locked linkage tester/designer tool
Infinitesimal locking rules: [Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribó, Rote 2006]

**Rule 1:**

- Acute angle
- Equal length
- Pinned together for positive time

**Rule 2:**

- Acute angle
- Equal length
- Ditto

**Example:**

- From rigid to strongly locked using Rule 1
- From strongly locked to rigid using Rule 2
3D knitting needles: locked if each end bar is longer than $\sum$ middle bars

Proof: draw ball $B$ centered at midpoint of middle bars, diameter $= \sum$ middle bars $+ \varepsilon$

$\Rightarrow$ middle vertices remain inside $B$,
end vertices remain outside $B$.
$\Rightarrow$ any motion could be augmented by an unknotted rope connecting two ends outside $B$.
$\Rightarrow$ straightening motion would untie trefoil knot. $\Box$

OPEN: minimum possible edge length ratio for which locked 3D chain exists?
- best example is $1:3+\varepsilon$ above

OPEN: any locked equilateral 3D chain? [Biedl et al.]
equilateral 3D chain self-weaving on line [E. Demaine]
equilateral unknotted closed chains? [M. Demaine]
equilateral trees? [E. Demaine; Poon]
equilateral chain of equal-width cylinders? [O'Rourke]