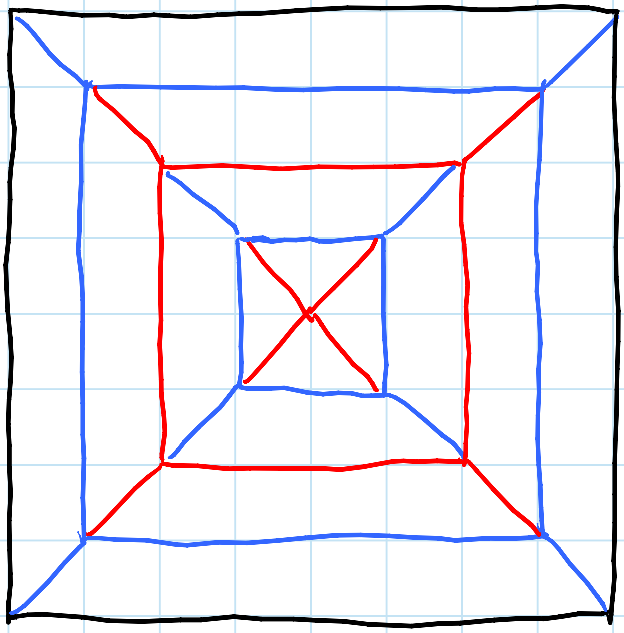
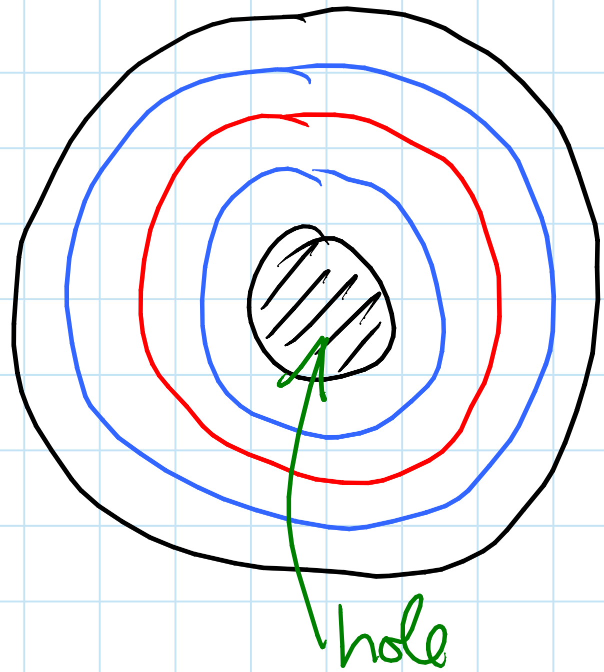
6.849	Lecture 22	Nov. 24. 2010

Pleat folding: [Albers @ Bauhaus, 1927–1928]

"Hyperbolic paraboloid"



Circular pleat:



Self-folding origami: physics finds equilibrium form automatically

→ complex 3D shape from simple creases

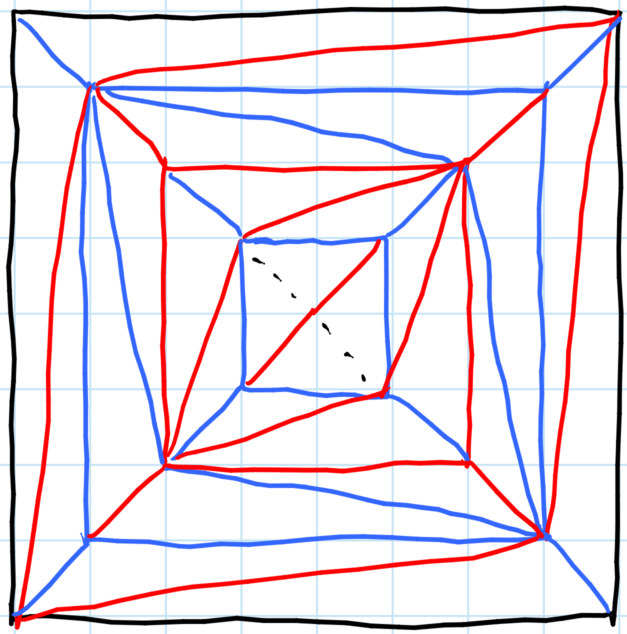
Forces: paper wants to
- stay flat where uncreased
- stay bent where creased

Creasing = plastic deformation beyond yield point, changing elastic memory of paper to nonzero fold angle

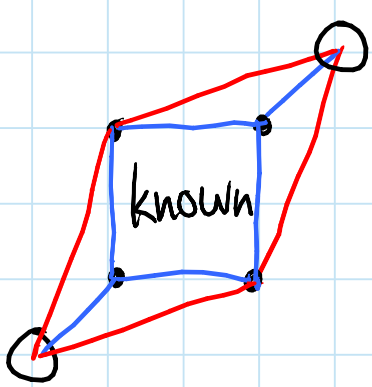
Is the "hyperbolic paraboloid" really (approximating) a hyperbolic paraboloid?

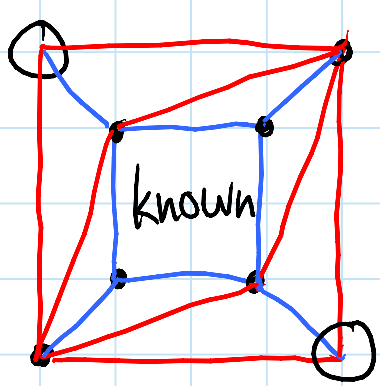
<u>Surprise</u>: "hyperbolic paraboloid" doesn't exist!
— impossible to <u>nontrivially</u> fold exactly that crease pattern ↳ fold angle ≠ 0, ±180°
[Demaine, Demaine, Hart, Price, Tachi 2009/2010]

<u>Good news</u>:
(& likely what's happening in RL)
possible to fold with extra creases & dropping a central crease:



<u>Proof</u>:
- fold central ⧄ <u>hinge</u> by some θ
- next layer out determined by intersections of 3 known spheres:





- intersection of 3 spheres can be computed by radical expression ($+, -, \cdot, /, \sqrt{\ }$) on center coords. & radii
  (though >150,000 terms!)

$\Rightarrow$ in theory, could specify exact 3D model by such an expression

- in practice, infeasible beyond second ring

- instead: use interval arithmetic to get coordinates within some range $[L, U]$
  - e.g. $[L_1, U_1] + [L_2, U_2] = [L_1 + L_2, U_1 + U_2]$
  - errors accumulate
- exists provided never take $\sqrt{[L, U]}$ with $L < \emptyset$ (then sphere intersection might not exist)
- also need to check no collision

- implemented in Mathematica
- checked for $n = 100$ rings & $\theta \in \{2°, 4°, \ldots, 178°\}$
- required precision depends on n (2,048 decimal digits suffice for $n = 100$)
- nonalternating triangulation doesn't fold to 180° (how much depends on n)

$\square$

So is it (approx.) a hyperbolic paraboloid?
- YES, very close, except at center
- great parabolic fit from just last 3 rings
- about 0.03% error at center
  ($\approx$ independent of n)

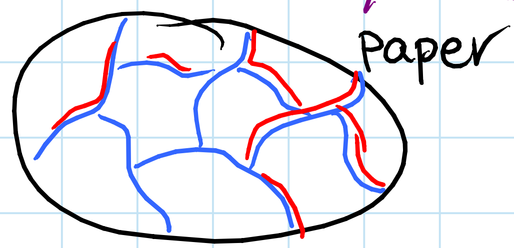
$\boxed{\underline{\text{OPEN}}}$: does triangulated folding exist for all n & $0 < \theta < 180°$?
- seems so, but lack tools to show

$\boxed{\underline{\text{OPEN}}}$: does circular pleat exist?
- conjecture yes

How paper folds between creases:

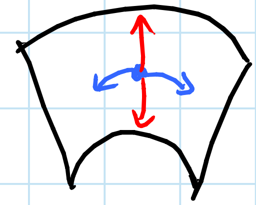
[Demaine, Demaine, Hart, Price, Tachi 2009/2010]

- require folding to be piecewise-C^2 & flat
- crease = C^1 discontinuity
- semicrease = C^2 discont.
- flat = intrinsically flat (AKA "developable")
  = zero curvature [Gauss's Theorema Egregium]



Gaussian curvature at a point (=0)
= product of two principal curvatures
↳ min & max curvatures

⇒ one is zero
- if both zero, planar point
- else parabolic point



Lemmas using differential geometry:

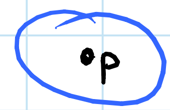
skip
- every proper semicrease is a line segment with endpoints on creases or boundary
- ⇒ no semivertices (except on creases)

- every smooth point lies on a rule segment with endpoints on creases or boundary, unique unless point has a planar neighborhood
- ⇒ ruled surface $c(s) + t \cdot \delta(s)$ around any point
  c: C^1, δ: $C^\emptyset$
- torsal: common tangent plane to each rule line
- points along rule line uniformly planar/parabolic

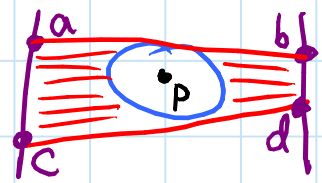
Polygonal ⇒ flat: if uncreased region's boundary folds to a 3D polygon, then entire region folds to a 3D plane

Proof: claim every point in region is planar

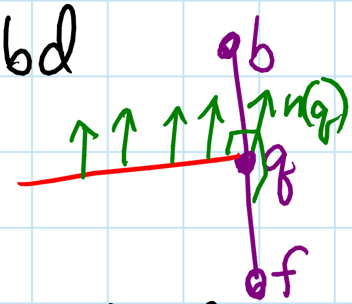
- consider parabolic point p



- a small neighborhood of p is entirely parabolic (by smoothness)
- take union of rule lines through those points



- ac & bd polygonal
- look at segment bf of bd
- $n(q)$ = normal at q on bf is perpendicular to bf & to q's rule line



- torsal ⇒ same normal along rule line

⇒ $n'(q)$ is perpendicular to rule line

derivative as q moves along bf

- also perp. to bf because all $n(q)$ are

⇒ $n'(q)$ has same direction as $n(q)$

⇒ $n'(q) = 0$

⇒ $n(q)$ constant

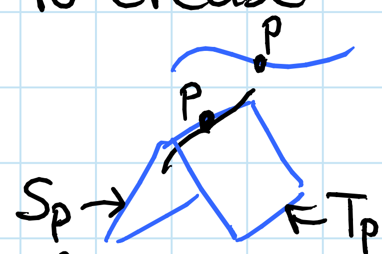
⇒ rule lines form planar region

- contradiction □

Straight creases stay straight:

geodesic crease with fold angle $\neq \pm 180°$
folds to 3D line segment

Proof: consider point p interior to crease



- surrounded by 2 sides
- compute tangent plane on each side: S_p, T_p
- different because fold angle $\neq \pm 180°$
- tangent vector p' along crease lies along $S_p \cap T_p$
- consider curvature vector p''
- crease is straight on unfolded paper & paper is locally flat S_p & T_p

$\Rightarrow$ crease should have zero curvature when projected onto S_p or T_p

$\Rightarrow$ p'' is perpendicular to S_p & T_p

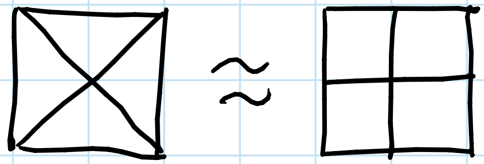
- $S_p \neq T_p \Rightarrow p'' = 0$

$\Rightarrow$ crease is a line segment. $\square$

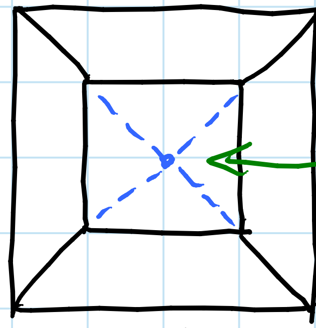
Nontrivial foldings of straight creases are $\approx$ rigid!

every interior face of straight crease pattern
↳ not touching boundary of paper
has polygonal boundary (creases $\rightarrow$ segments)
& thus is planar in 3D i.e. rigid
(boundary faces might not be rigid)

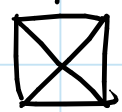
Back to "hyperbolic paraboloids":

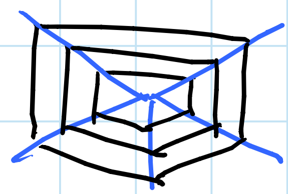
Center is bad:  rigid

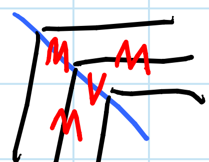
- can fold one crease but not both (for nontrivial folding)

Any ring is bad: 

extensions meeting

⇒ induces folding of  again

More generally: two rings bad if diagonal extensions meet 

- must have local mountain/valley assignments like 

(or reverse)

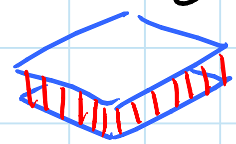
⇒ diagonals in ring are all M or all V

- induces all-M or all-V folding of 

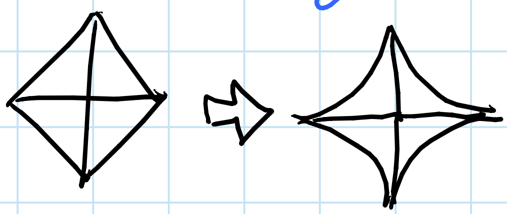
- impossible

OPEN: what is the maximum volume whose surface is a folding of a teabag

doubly covered square



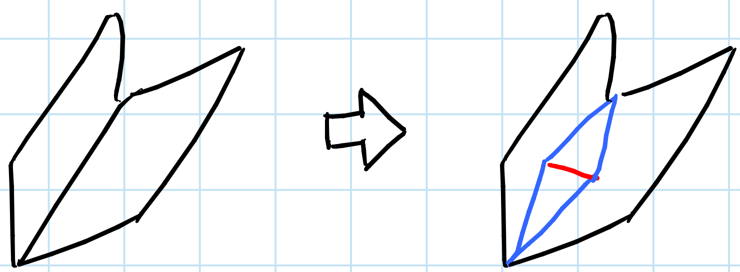
e.g.



4-hypar

Inflation:

- every generic convex polyhedron can be folded to increase volume [Bleecker 1996]
  "by simultaneously delivering karate chops to the edges of the polyhedron"



valley

- every polyhedron can be folded to increase volume [Pak 2006]

⇒ limit not polyhedral

Curved creases

[Huffman 1970s–1990s]
[Demaine, Demaine, Koschitz 2010]
+ Huffman family