

Origami terminology:

Piece of paper = 2D polygon (most often)
with distinguished top/bottom sides

Crease = line segment or curve on paper

Crease pattern = collection of creases

= planar graph

drawn on paper

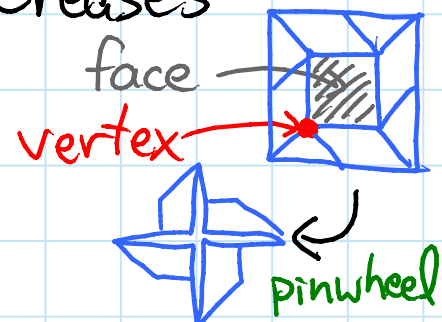
Folded state = finished origami

— unfolding → crease pattern

Flat folding = folded state lying in a plane

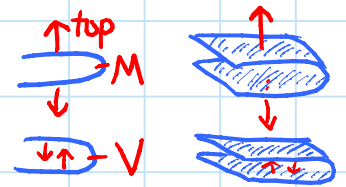
— call its crease pattern flat foldable

(must use all creases)



Mountain crease = bottom sides touch

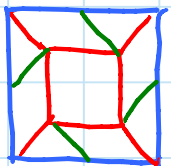
Valley crease = top sides touch



Mountain-valley assignment = which creases

in crease pattern are mountain/valley

origami notation: $\cdots\cdots\cdots / \cdots\cdots\cdots$



Mountain-valley pattern = crease pattern

+ mountain-valley assignment

Example: crumple up a piece of paper

Simple fold: fold along a single line,
by $\pm 180^\circ$ (mountain/valley)
- choice of how many layers to fold

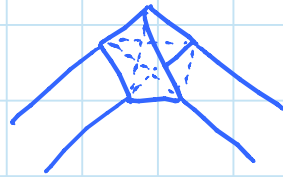
One-layer simple fold = just top or bottom
All-layers simple fold = all the way through



TODAY is mostly about simple folds

Strip: long narrow rectangle

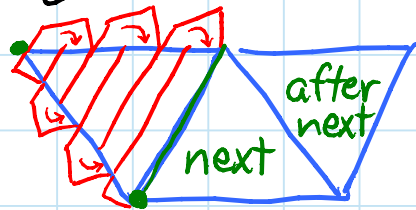
Example of nonsimple strip folding:
tie a knot



Folding any shape: [Demaine, Demaine, Mitchell 2000]
(a.k.a. silhouette [Bern & Hayes 1998] / gift wrapping [Akiyama/Gardner])

Every connected union of polygons in 3D,
each with a specified visible color (on each side),
 can be folded from a sufficiently large piece of bicolor paper of any shape (e.g., square).

Proof: fold paper down to long narrow strip (!)
- triangulate the polygons
- choose a path visiting each triangle at least once
- cover each triangle along the path
by zig-zag parallel to next edge,
starting at opposite corner:

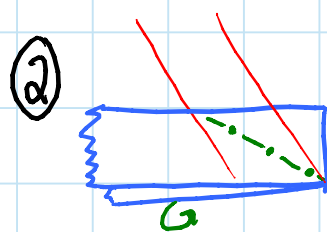


choose parity of zig-zag to arrive
at correct corner for next triangle

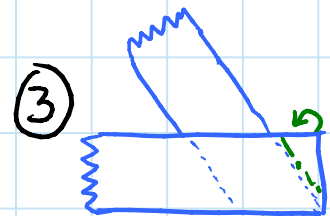
- turn gadget implements zig-zags & vertex turns:



perpendicular mountain



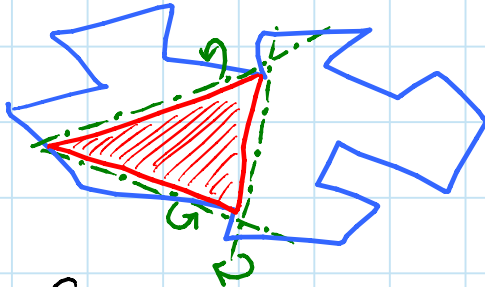
fold bottom layer



hide excess
(many folds)

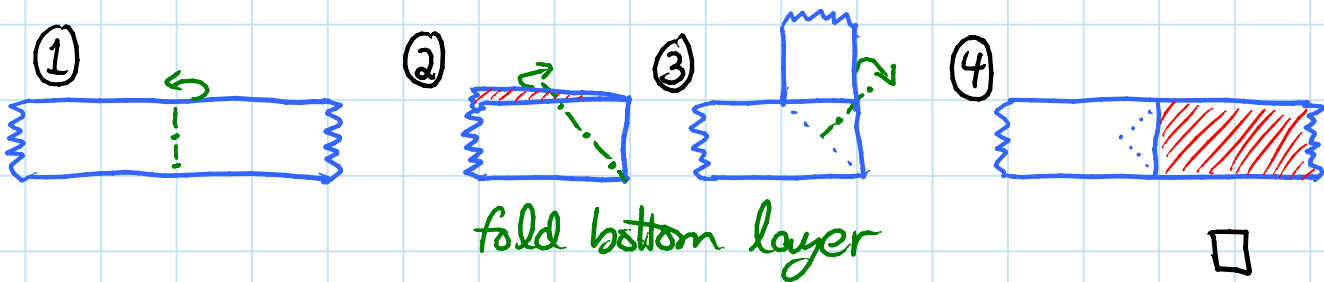
Proof of folding any shape: (cont'd)

- hide excess paper underneath each triangle:
(more generally, can hide under any convex polygon)



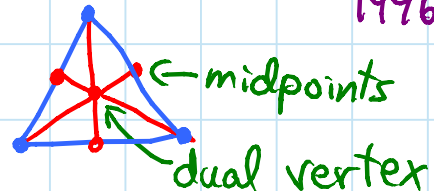
repeatedly mountain fold
along lines extending
desired edges

- if paper is unicolor (or don't care)
can use valley folds $\Rightarrow$ simple folds
 - if mountain folds, might collide with
other Δ s $\Rightarrow$ not really simple folds
(but still works as origami fold)
- color-reversal gadget along transition
between triangles of opposite colors:

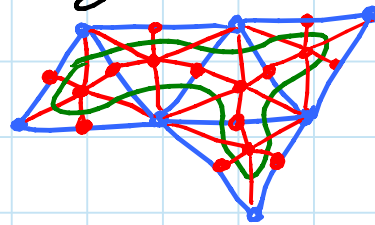


Pseudo-efficiency: if allowed to start with any rectangle of paper, then can achieve
 $\text{area}(\text{paper}) = \text{area}(\text{surface}) + \varepsilon$ for any $\varepsilon > 0$

Proof: construct Hamiltonian refinement of triangulation: [Arkin et al. 1996]

– cut each  into 

– walk around spanning tree of original dual:



- now visit each triangle exactly once
- wastage from turns $\rightarrow 0$ with strip width. $\square$

Alternate proof from class: (should work)

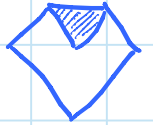
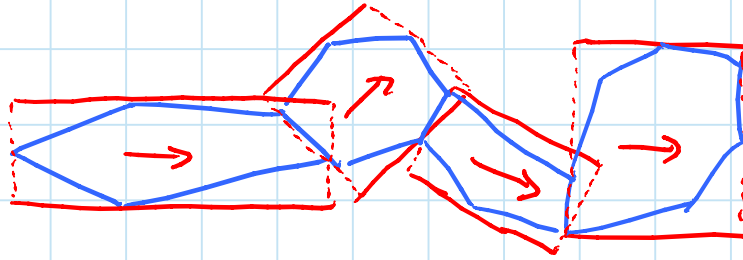
- visit Δ s in any order, but when traversing from one to next, go directly (without covering (much of) intervening Δ s)
- wastage is $\approx \text{strip width} \cdot \sum \text{diameter}(\Delta)$
 $\rightarrow 0$ as strip width $\rightarrow 0$

OPEN: pseudopolynomial upper bound? lower bound?

(uncovered:)

Seam placement: can place seams (visible creases/
paper boundary) as desired, provided regions
between seams are convex

— idea: vary strip width, use hide gadget



OPEN: what seam placements are possible?

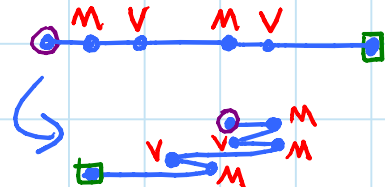
1D flat folding: [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2004]

Piece of paper = line segment

Crease = point on paper

Flat folding lies on a line

All crease patterns are flat foldable:
zig-zag / accordion fold



Not all mountain-valley patterns:

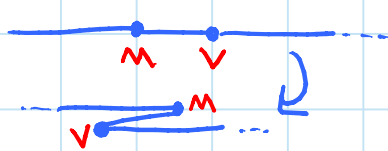


Two folding operations: (both simple)

① end fold if end length $\leq$ neighbor



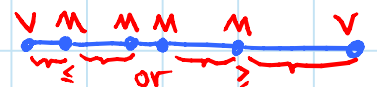
② crimp two consecutive creases
if length between $\leq$ both neighbors
& one mountain, one valley



Characterization:

mountain-valley pattern is flat foldable
 $\Leftrightarrow$ there's a sequence of crimps & end folds

Tool: Mingling: for any maximal sequence of M's or V's,
adjacent V or M or end on at least one side
is nearer than adjacent M or V:



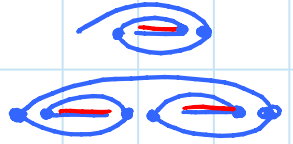
OPEN: proof without mingling?

[Solved by Akitaya,
Brunner, Demaine,
Hendrickson, Luo,
Tockman - TSM 2023]

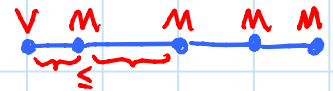
1D flat folding (cont'd)

Proof of characterization:

- flat foldable $\Rightarrow$ mingling:
 - sequence of M's or V's form a spiral (or double spiral)
 - at least one end must be short
- mingling $\Rightarrow$ end fold or crimp possible

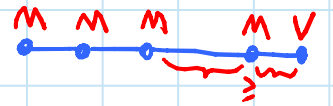


for each maximal sequence of M's or V's
write (if "left mingling"



[otherwise

& write) if "right mingling"



] otherwise

$\Rightarrow$ (] or [) or (

-)($\Rightarrow$ crimp

- leading (/ trailing) $\Rightarrow$ end fold

- if neither: [) [) ... [) (] (] ... (]

$\xrightarrow{\text{crash}}$

- crimp/end fold preserves flat foldability

- take flat folding

- move some layers out of crimp

$\Rightarrow$ could start with crimp



- induct $\Rightarrow$ sequence of crimps & end folds $\Rightarrow$ flat foldable again. $\square$

2D map folding: [Arkin et al. 2004]

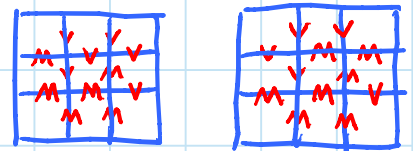
- ↳ rectangular paper with axis-parallel creases
- again every crease pattern is flat foldable:
zig-zag in x then y



OPEN: characterize flat-foldable mountain-valley patterns — even $3 \times n$! [Edmonds 1997]
($2 \times n$ solved by Demaine, Liu, Morgan [Morgan - MEng 2012])

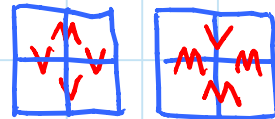
Simple folds are not as powerful in 2D:

(in contrast to 1D, where we can simulate crimp/end folds)



Characterization of simple foldability of maps:

- if simply foldable, must be a uniform horizontal/vertical line
↳ all M or all V
- crossing vert./horiz lines must switch $M \leftrightarrow V$ here:
 - local 2×2 patterns:



& rotations

- ⇒ all uniform horizontal lines must be folded before any vertical lines become uniform, etc.
- ⇒ sequence of 1D problems on current uniform lines
- linear-time algorithm — $O(mn)$ for $m \times n$ map —
by maintaining uniformity for each line,
crimpability & end foldability on lengths