6.849: Geometric Folding Algorithms
Fall 2012 — Prof. Erik Demaine, Jayson Lynch

Problem Set 4

Due: Thursday, October 11th, 2012

We will drop (ignore) your lowest score on any one problem.

Problem 1. Prove that, for any polygon with n vertices, its straight skeleton has $O(n)$ vertices, edges, and faces (as mentioned in lecture). You may assume any other properties of the straight skeleton mentioned in lecture.

Solution: We will look at the graph created by the union of the polygon and the straight skeleton by induction. Let us consider how many edges, vertices, and faces it contains.

Faces: The method of construction by shrinking edges leads to a correspondence between every edge in the original polygon and two faces in the new graph. Since there are n edges in a polygon with n vertices, the new graph will have $2n$ faces.

Vertices: A splitting event only occurs when there is a reflex angle in the polygon. The number of events where an edge or face shrink to zero are bounded by the number of edges in the region. An edge can only be part of two regions. Thus the number of vertices is bounded by twice the edges plus the number of reflex angles.

Edges: Since the resulting graph is planar, we can use Euler's Formula, $V - E + F = 2$, and the bounds for the number of faces and vertices to bound the number of edges.

A more general proof about the linear complexity of straight skeletons can be found in Aichholzer and Aurenhammer's Straight Skeletons for General Polygonal Figures in the Plane.

Problem 2. Design and fold (but do not cut) a fold-and-cut model using the straight-skeleton method. Email us a copy of your design (including crease pattern, in vector format) and submit a folded one with your pset. We highly recommend that you use a vector drawing program that can compute accurate intersections, such as Inkscape (free), Cinderella (mostly free), Adobe Illustrator (commercial), AutoCAD (commercial), or Rhino3D (commercial).
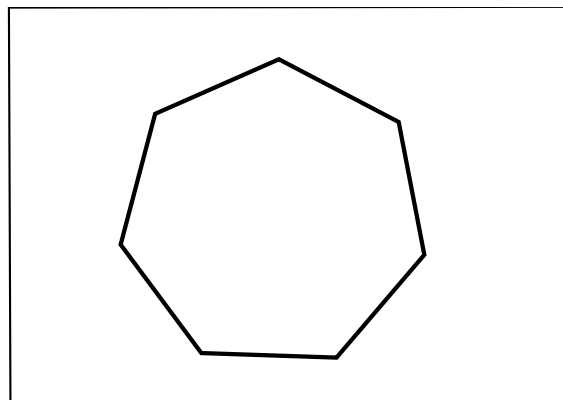
Solution: There is no one correct answer.

Problem 3. Which of the following graphs can be cut by all-layers simple folds and one complete straight cut? For each, draw a folding sequence or argue why there is none.
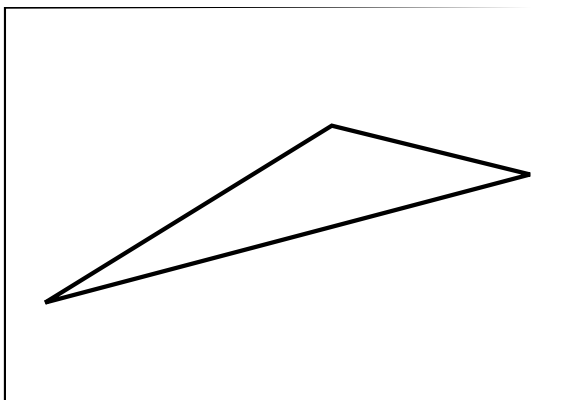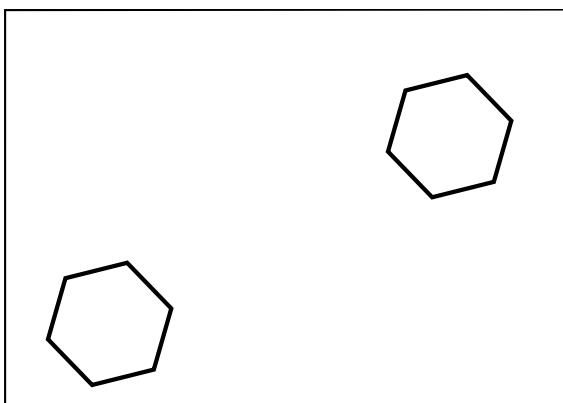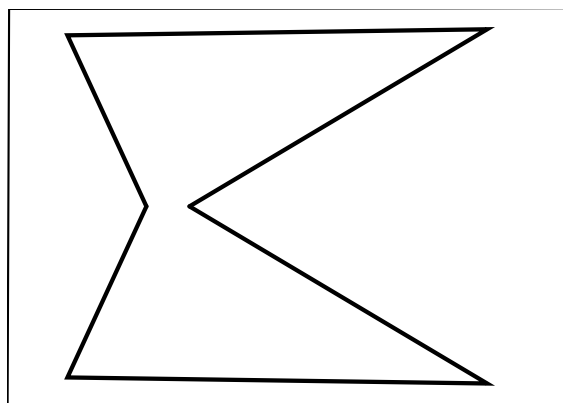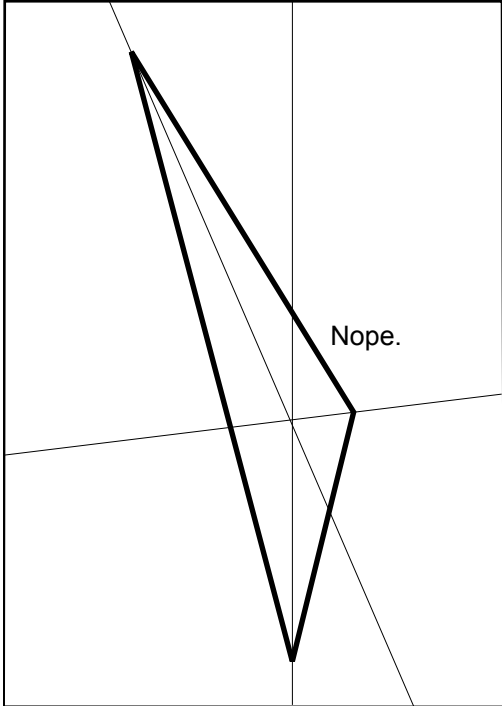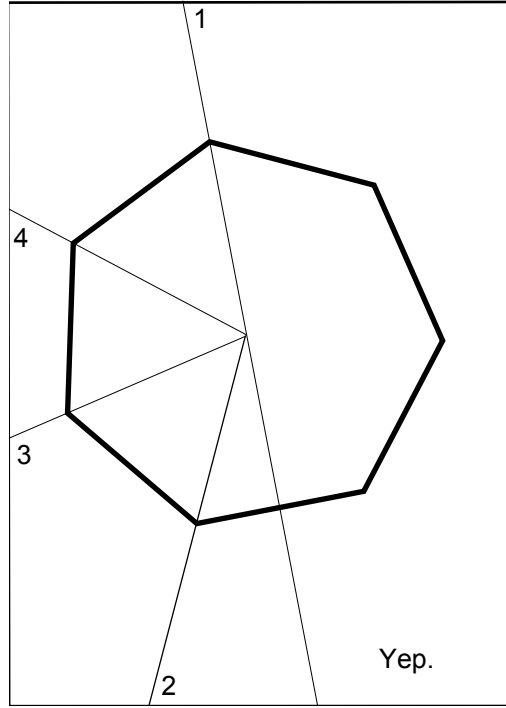
(a)

(b)

(c)

(d)

Solution: Refer to the figure below.

(a) This cannot be constructed by all-layers simple folds. All the edge lengths are different and thus there is no axis of symmetry.

(b) This can be constructed by all-layers simple folds. It is a regular heptagon and thus can be folded in half. After that the angles can be bisected.

(c) This can be constructed by all-layers simple folds. First, note the hexagons are reflections over a horizontal and vertical line. In a single polygon this is a problem, as two reflections may not be decomposable into one, and any fold not along symmetry intersecting the polygon will move lines over non-lined paper. Here we can fold the edges down, so the hexagons lie along a strip. Now fold the strip so the hexagons lie reflected across from each other, and then fold again so they lie on top of each-other. It is easy to see a single hexagon can be constructed by all-layers simple folds.

Another solution is provided in the diagram below. This involves two folds to effectively rotate the hexagons relative to each other, allowing them to be folded directly onto each other.
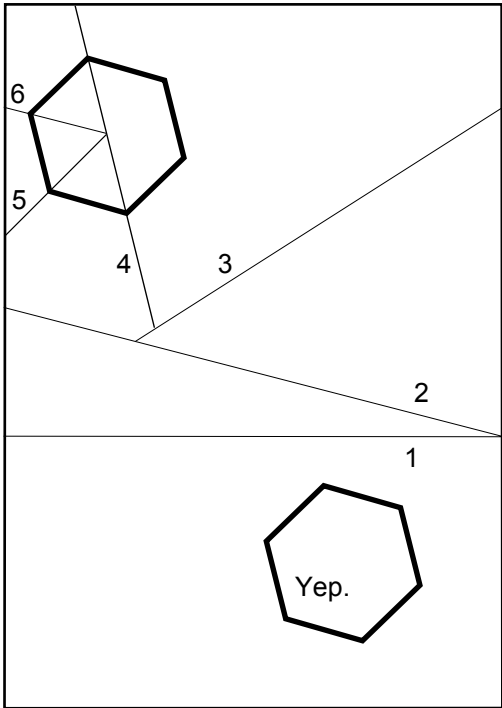
(d) This cannot be constructed by all-layers simple folds. The only initial line of symmetry is down the middle. After the initial fold, we have most of a triangle and may notice that the bisectors of the two remaining angle intersect the opposite sides at angles other than 90°.
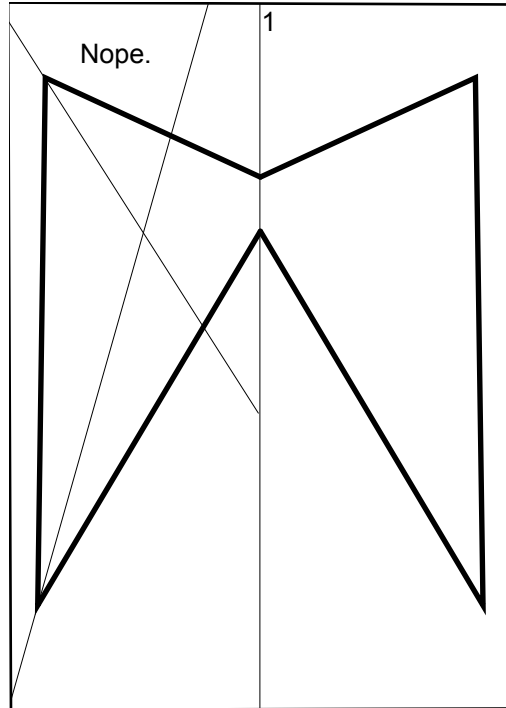
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(a)

(b)

(c)

(d)

Nope.

Yep.

Yep.

Nope.