

**Problem Set 3 Solutions**

*Due: Tuesday, October 2nd, 2012*

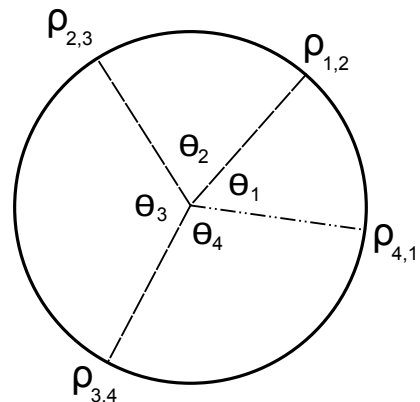
**We will drop (ignore) your lowest score on any one problem.**

**Problem 1.** Design and fold a piece of origami using Tomohiro Tachi’s Freeform Origami software or Alex Bateman’s Tess. Submit the physical folded piece in class with your name on it, and send a digital copy of the crease pattern to 6849-staff@csail.mit.edu.

- Freeform Origami can be downloaded from <http://www.tsg.ne.jp/TT/software/#ffo>  
 (Windows only. If you do not have a Windows machine, use a Windows Athena cluster, e.g., in building 37-3.)
- Tess can be downloaded from <http://www.papermosaics.co.uk/software.html>  
 (Requires Perl/Tk, or use the Windows binary.)

*Solution:* There is no one correct answer.

**Problem 2.** Given a flat-foldable degree-4 vertex, we can represent its configuration in 3D space by the four angles between the creases  $(\theta_1, \theta_2, \theta_3, \theta_4)$  and by four *folding angles*  $(\rho_{1,2}, \rho_{2,3}, \rho_{3,4}, \rho_{4,1})$ , as shown on the right. We measure a folding angle between  $-180^\circ$  and  $180^\circ$ : a folding angle of 0 indicates no folding, while a folding angle of  $\pm 180^\circ$  indicates a mountain/valley flat fold. Prove that, for any 3D configuration of a flat-foldable degree-4 vertex,  $|\rho_{1,2}| = |\rho_{3,4}|$  and  $|\rho_{2,3}| = |\rho_{4,1}|$ . (For the mountain-valley assignment in the figure,  $\rho_{1,2} = \rho_{3,4}$  and  $\rho_{2,3} = -\rho_{4,1}$ .)



**Hint:** Use spherical trigonometry.

*Solution:* Choose points along the creases all at distance 1 from the vertex of the fold. These points form a spherical quadrilateral on the surface of a sphere of radius 1, whose edge lengths are  $\theta_1, \theta_2, \theta_3, \theta_4$ , and whose vertex angles are  $\rho_1, \rho_2, \rho_3, \rho_4$ . Call the interior diagonal of this quadrilateral  $x$ . Then, by the Spherical Law of Cosines, we have:

$$\cos x = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{2,3}, \quad (1)$$

$$\cos x = \cos \theta_1 \cos \theta_4 + \sin \theta_1 \sin \theta_4 \cos \rho_{4,1}. \quad (2)$$

Further, by Kawasaki’s Theorem, we know:

$$\theta_1 + \theta_3 = \theta_2 + \theta_4 = \pi. \quad (3)$$

From trigonometry, we know:

$$\cos(\pi - \theta) = -\cos \theta, \tag{4}$$

$$\sin(\pi - \theta) = \sin \theta. \tag{5}$$

Combining Equations 1 through 6, we get:

$$\cos x = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{2,3} = \cos \theta_1 \cos \theta_4 + \sin \theta_1 \sin \theta_4 \cos \rho_{4,1} \tag{6}$$

$$\cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{2,3} = \cos(\pi - \theta_3) \cos(\pi - \theta_2) + \sin(\pi - \theta_3) \sin(\pi - \theta_2) \cos \rho_{4,1} \tag{7}$$

$$\cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{2,3} = (-1)(-1) \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{4,1} \tag{8}$$

$$\cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{2,3} = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \rho_{4,1} \tag{9}$$

$$\cos \rho_{2,3} = \cos \rho_{4,1} \tag{10}$$

Because we know the values of our fold angles are restricted to the range  $[-\pi, \pi]$ , we know  $\rho_{2,3} = \pm \rho_{4,1}$ . A similar argument using different angles and spherical triangles gives the other pair of angles.

**Problem 3.** Make a cool maze-folding design using the following webapp:

<http://erikdemaine.org/fonts/maze/>

Email us a link to your design using the “link to this view” feature. You do not need to fold this piece. If you decide to try folding it, we will expect a much less complex design.

*Solution:* There is no one correct answer.

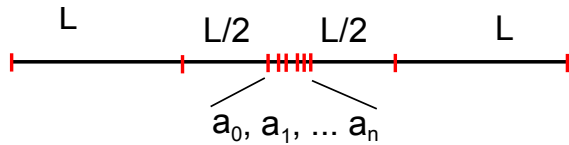
**Problem 4.** Prove that the following problem is NP-hard:

Given a 1D piece of paper with a 1D crease pattern, find the subset of creases to fold that produces the smallest length of the resulting flat folding.

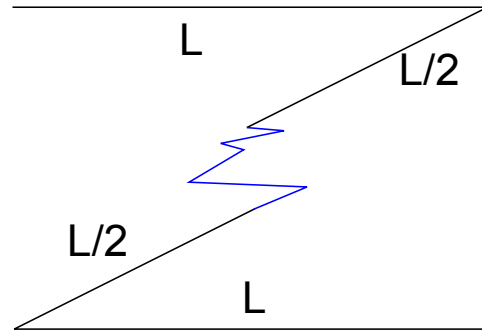
Note that not all creases need to be folded and there is no mountain-valley assignment.

**Hint:** Reduce from Partition.

*Solution:* We prove NP-hardness of the above problem by reduction from Partition. The Partition problem asks, given a multi-set  $S = \{a_0, a_1, \dots, a_n\}$ , whether there exists a partition into two subsets  $S_1$  and  $S_2$  such that the sum of the elements of  $S_1$  equals the sum of the elements of  $S_2$ . We construct an instance of the folding problem with creases placed with the following distances between them:  $\{L, L/2, a_0, a_1, \dots, a_n, L/2, L\}$ , where  $T = \sum_{i=0}^n a_i$  and  $L > 2T$ .



Unfolded State



Folded State

Because of the edges of length  $L$ , it is impossible to fold the paper smaller than length  $L$ . To achieve this minimum length, both bars of length  $L$  must lie on top of each other, and thus the bars of length  $L/2$  must have endpoints at  $0, L/2$  or  $L/2, L$ . This property can occur only if the sections in the middle come back to the same location they start at, i.e., position  $L/2$ . Now, when we decide not to fold a crease, we have lengths continuing in the same direction, while when we fold the crease, they then move in the opposite direction. Thus, to have the start and end of this subsection of the paper return to the same place, we must be able to pick lengths traveling to the right and lengths traveling to the left such that the sum of the lengths in each group is the same. (You may have noticed a similarity to Kawasaki's Theorem, because in that case the paper needed to meet itself to be circular.)

Thus, we can instantiate the Partition problem in the folding problem. If the minimum length is  $L$ , then we know that there is a partition of the sets. In addition, if we know the partition, we can construct a folding of length  $L$  by folding a crease whenever the next corresponding segment is partitioned into a different set than the previous one, and not folding a crease when they are both in the same set. Therefore the folding problem is NP-Hard.