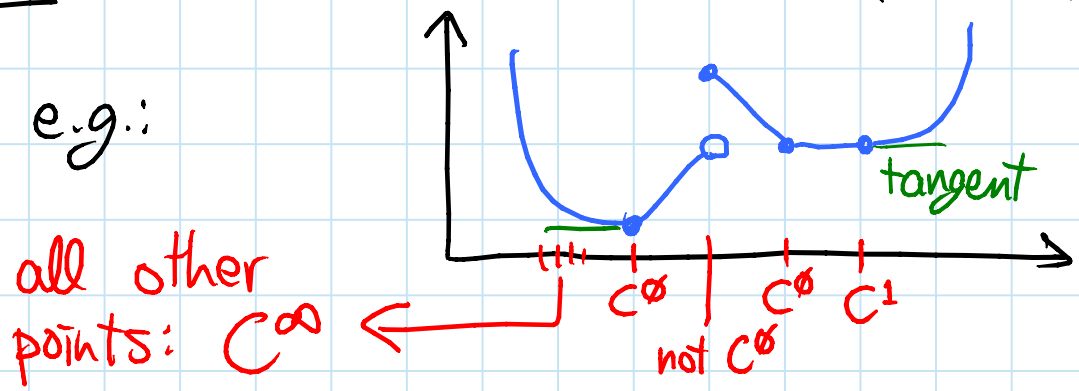
6.849	Class 9	Oct. 4, 2012

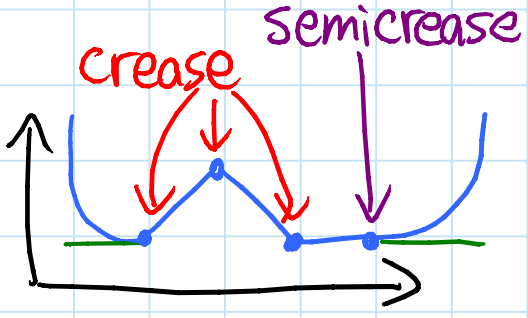
- Triangulated hypars:
  - OPEN: does alternating triangulation work for arbitrary n?
  - OPEN: do triangulations mixing alternation/not do medium well?
  - OPEN: what about regular k-gons?
    - issue: how to get started?

- Smoothness: (↗ no gaps (undefined) & no jumps)
  - C^0 = continuous function f
  - C^1 = continuous derivative f'
  - C^2 = continuous second deriv. f''
  - C^∞ = continuous derivatives f, f', f'', ...

e.g.:



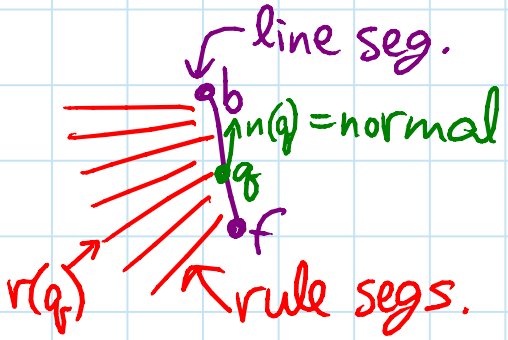
  - well-behaved folding = C^0 & piecewise-C^2



- **Polygonal ⇒ flat proof:**
  - $n(q) \perp bf$
  - $n(q) \perp r(q)$

  } definition of normal

  $\Rightarrow n'(q) \perp bf$
  & $n'(q) \perp r(q)$



  ... because $n(q+\varepsilon) \approx n(q) + \varepsilon \cdot n'(q)$ [Taylor series]

  (move q along bf)

  $\Rightarrow 0 = n(q+\varepsilon) \cdot bf \approx \underbrace{n(q) \cdot bf}_{=0} + \varepsilon \cdot \underbrace{n'(q) \cdot bf}_{=0}$ $\Rightarrow$

  & $0 = n(q+\varepsilon) \cdot r(q+\varepsilon)$

  $\approx \underbrace{n(q) \cdot r(q)}_{=0} + \varepsilon [\underbrace{n(q) \cdot r'(q)}_{=0 \text{ by torsal}} + \underbrace{n'(q) \cdot r(q)}_{=0}] + \varepsilon^2 \ldots$ $\Rightarrow$

  ⋮

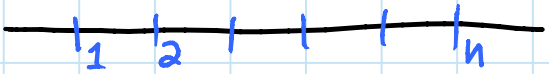
- Mathematical vs. real paper

  what does it all mean?

  - real paper might stretch/shear slightly
  - real paper might add many small creases

    (too slight to see)

  PROJECT: investigate!

- Pleat folding algorithms:
  [Cardinal, Demaine, Demaine, Imahori, Ito, Kiyomi, Langerman, Uehara, Uno – G&C 2011]
  - model: 1D paper 

    n uniformly spaced creases
    some layers simple folds
    unfold previous fold (for free)
  - look at resulting string of M/V/unfolded
  - ≈ lg n folds makes "dragon curve":
    [Heighway, Banks, Harter ≈ 1967]
    M M V M M V M M V M V V M V V ...
    (& ≈ n variations)
  - MM...M can be folded by $\frac{3}{2}\lg^2 n$ folds
    1. fold in half until 3 creases left [xxx]
    2. fold them as M's: unfold all [MMM]
       → ?MMM?VVV?MMM?VVV?...
    3. fold VVV's on top of each other by rep. folding middle M of middle MMM
       → [M?VVV?M] → [MMMMMMM]
    4. fold 5 middle creases as M: unfold all
       → VM?MMMMMMM?MVVVVVM? etc.
    5. etc. O(lg n) rounds (doubling M reps.)
       · O(lg n) folds/round □

$\Rightarrow$ MVMV… can be folded by $3\lg^2 \frac{n}{2}$ folds

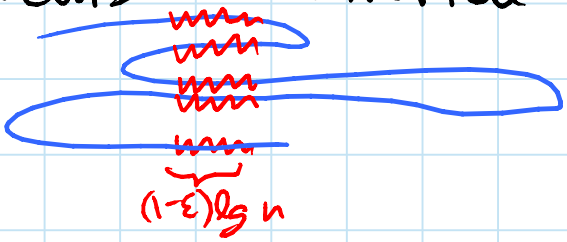
- MM…M requires $\approx \frac{1}{4}\lg^2 n / \lg\lg n$ folds
  - let $f(n) = \#$ folds for $M^n \le \frac{3}{2}\lg^2 n$
  - $\Rightarrow$ fold $n/f(n)$ creases at once on average
  - $\Rightarrow$ some fold does $\ge n/f(n)$ creases at once
  - half of these creases become Valleys
  - need $\approx \lg n/f(n)$ folds to set this up
  - need $\approx \lg \frac{1}{2} n/f(n)$ folds to set up Mountain folding these valleys
  - etc.
  - $\Rightarrow$ need $f(n) \gtrsim \log_{f(n)} n \cdot \lg \frac{n}{f(n)} = \Omega\left(\frac{\lg^2 n}{\lg\lg n}\right)$
    ($\le \frac{3}{2}\lg^2 n$; $\le \frac{3}{2}\lg^2 n$) □

OPEN: $\Theta\left(\frac{\lg^2 n}{\lg\lg n}\right)$ or $\Theta(\lg^2 n)$ optimal?

- MVMV… requires $\approx \frac{1}{4}\lg^2 n / \lg\lg n$ folds too

- most M/V strings require $\approx n/\lg n$ folds
  - k folds $\Rightarrow \le (2n)^k$ choices (M/V ↑ ↑ where)
  - another $\approx 4^k$ choices for how to place $\le k$ unfolds among k folds } Catalan
  - $k = \frac{n}{3+\lg n} \Rightarrow 4^k(2n)^k = (8n)^k = 2^{(3+\lg n)\cdot k} = 2^n$
  - 2^n M/V strings □

- all M/V strings can be folded by $(4+\varepsilon)n/\lg n$
  - divide string into $\Theta(n/\lg n)$ chunks each of length $(1-\varepsilon)\lg n$
  - $2^{(1-\varepsilon)\lg n} = n^{1-\varepsilon}$ chunk values
    $\Rightarrow$ average chunk repeated $n^{\varepsilon}/\lg n$ times
  - for each chunk value:
    1. fold repetitions together
    2. make their $\Theta(\lg n)$ folds
    3. fix creases messed up in 1
    4. recurse on inverted half



$n^{1-\varepsilon} \cdot (n^{\varepsilon}/\lg n + \lg n + n^{\varepsilon}/\lg n) \cdot 2$

↑ geometric series

OPEN: explicit (family of) examples requiring $\Theta(n/\lg n)$ folds?

OPEN: complexity of shortest fold sequence of given M/V string?
- known: in EXPTIME

- Let's fold hypars!