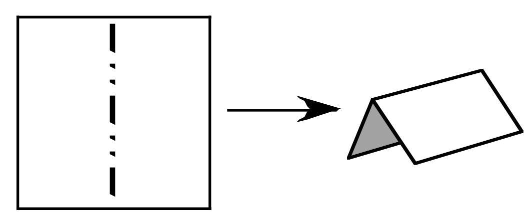
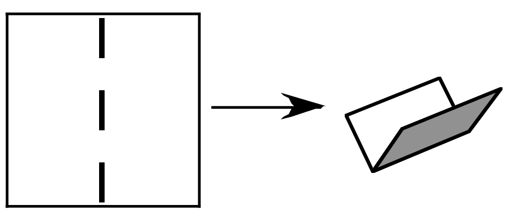
Maekawa and Kawasaki Revisited and Extended

Thomas C. Hull
Western New England College

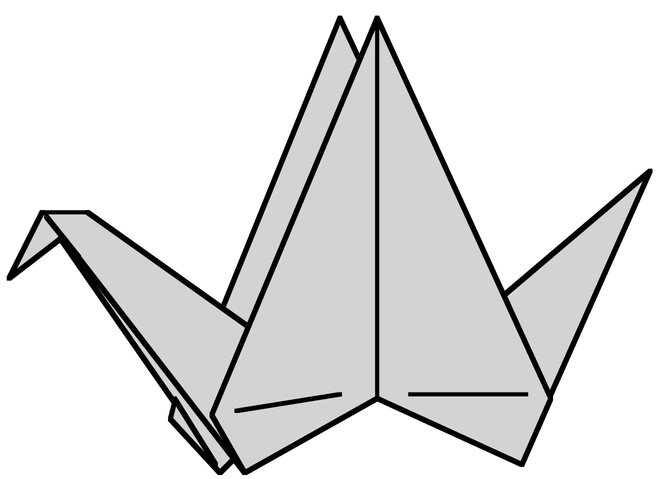
What is a flat origami?

- An origami model whose finished result can be pressed in a book without crumpling or adding new creases.

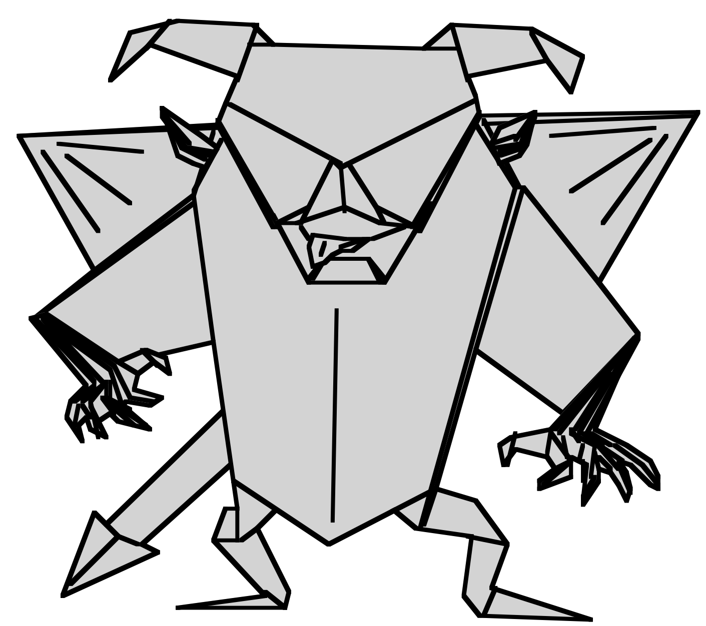
- Implies that all creases are **straight lines**.

- All creases are either **mountains**  or **valleys** 

- Examples:



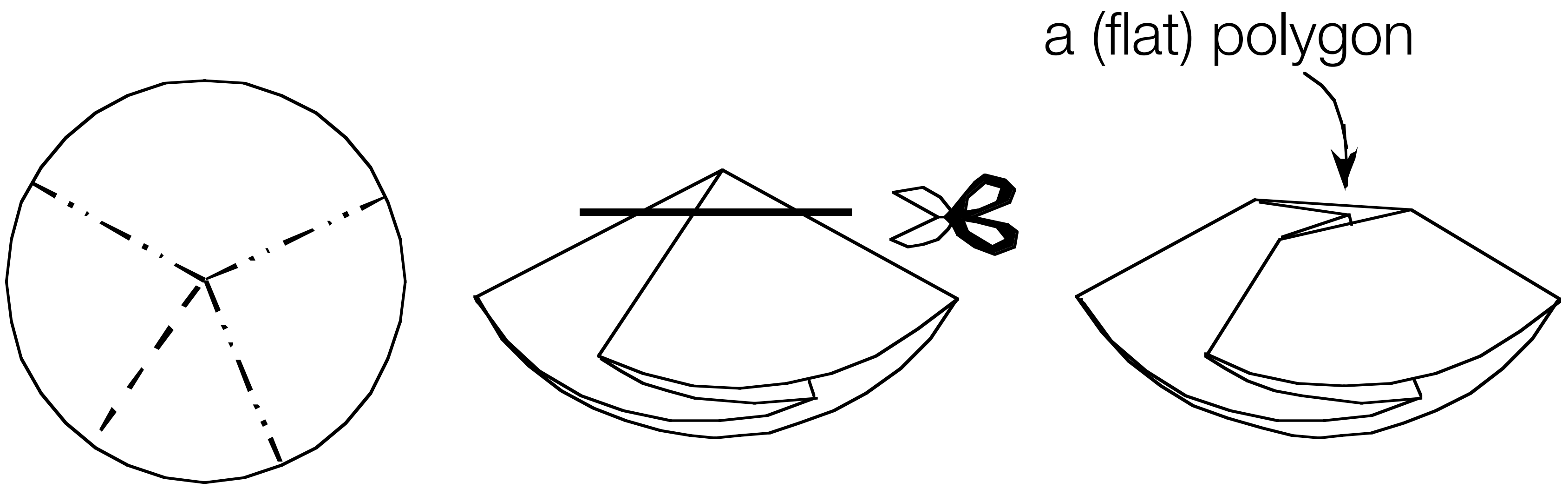
Traditional flapping bird (crane)



Jun Maekawa's Devil

Flat vertex folds

- Looking at a single vertex in a flat origami crease pattern.
- The vertex is in the paper's interior (not on the boundary of the paper).
- **Maekawa's Theorem:** The difference between the number of mountain and valley creases in a flat vertex fold is always two. (| M - V | = 2)

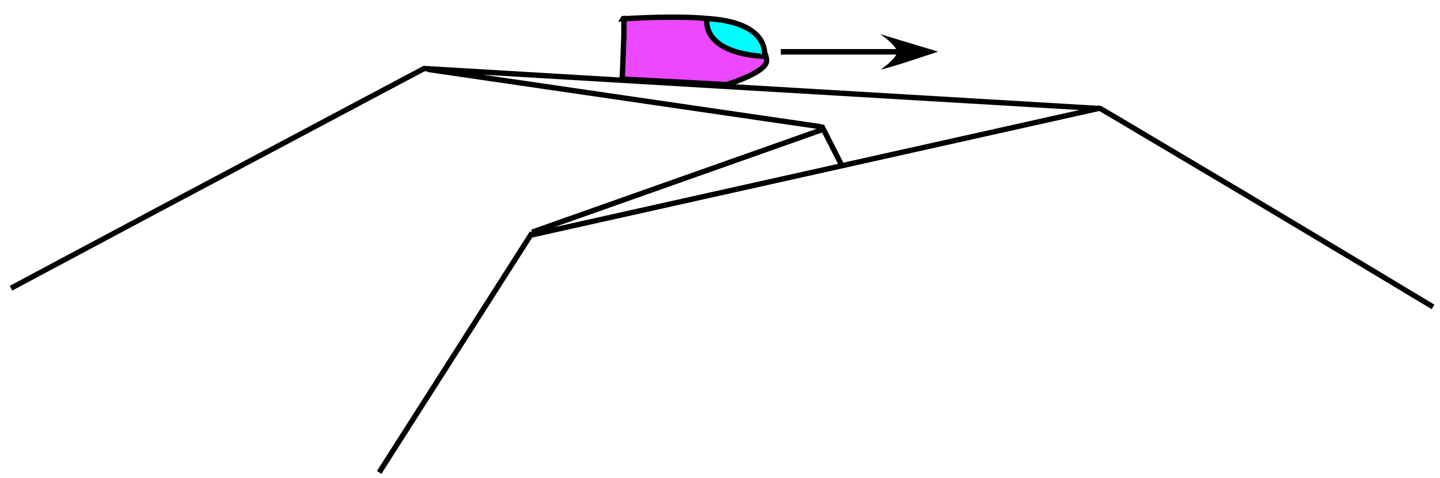


Flat vertex folds

- Looking at a single vertex in a flat origami crease pattern.

- The vertex is in the paper's interior (not on the boundary of the paper).

- **Maekawa's Theorem:** The difference between the number of mountain and valley creases in a flat vertex fold is always two. (| M - V | = 2)

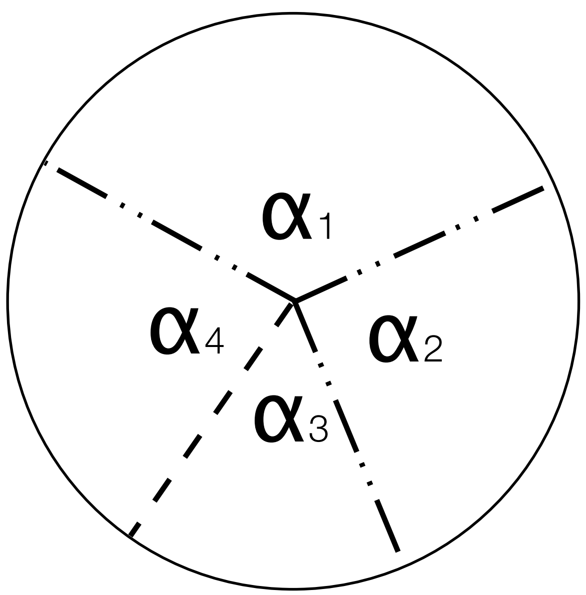


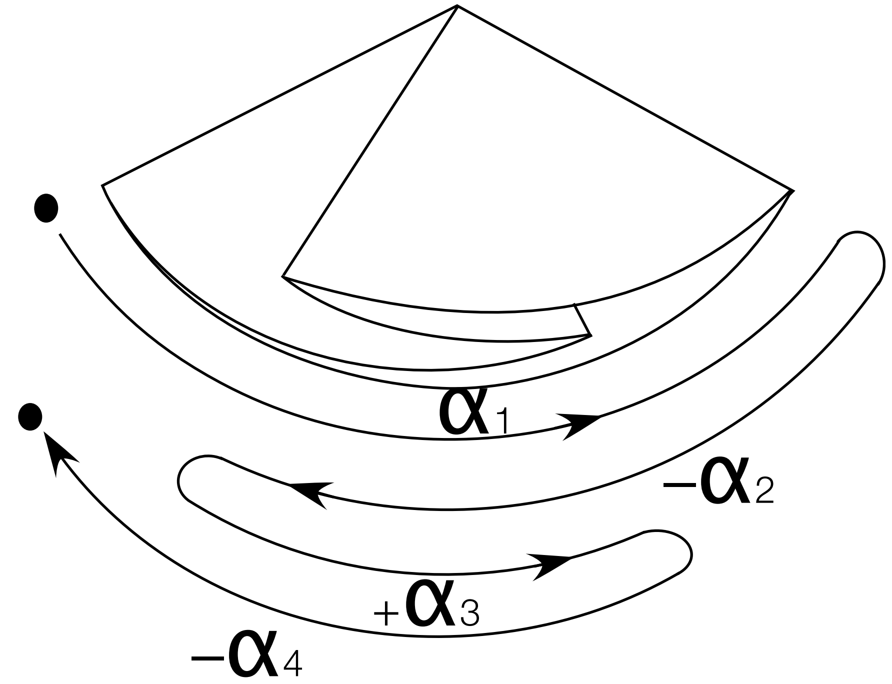
The monorail rotates 180° at each M and -180° at each V.
Thus 180 M – 180 V = 360,
or M – V = 2.

Flat vertex folds

- **Kawasaki's Theorem:** A collection of creases meeting at a vertex are flat-foldable if and only if the sum of the alternate angles around the vertex is π.

Proof of ⇒: Walk around the vertex, starting at a crease on the flat-folded object.





So... $\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \ldots - \alpha_{2n} = 0$

add to this $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots + \alpha_{2n} = 2\pi$

and you get $2\alpha_1 + 2\alpha_3 + \ldots + 2\alpha_{2n-1} = 2\pi$

that is, $\alpha_1 + \alpha_3 + \ldots + \alpha_{2n-1} = \pi$

History of these Theorems

- Kawasaki & Maekawa discovered these in the early 1980s.
  Reference: *Top Origami* by Kasahara & Takahama, 1985 (Japanese version of Origami for the Connoisseur).

- Justin discovered both of these in the 1980s as well.
  Reference: *British Origami Magazine*, 1986.

History of these Theorems

- Kawasaki & Maekawa discovered these in the early 1980s.
  Reference: *Top Origami* by Kasahara & Takahama, 1985 (Japanese version of Origami for the Connoisseur).

- Justin discovered both of these in the 1980s as well.
  Reference: *British Origami Magazine*, 1986.

- Also, it's not clear if Kawasaki originally saw the sufficiency direction of his Theorem, although Justin clearly did.

- Other people, like Huffman (1976) and Husimi (1979) discovered the degree 4 case (only) of Kawasaki.

History of these Theorems

- Kawasaki & Maekawa discovered these in the early 1980s.
  Reference: *Top Origami* by Kasahara & Takahama, 1985 (Japanese version of Origami for the Connoisseur).

- Justin discovered both of these in the 1980s as well.
  Reference: *British Origami Magazine*, 1986.

- Also, it's not clear if Kawasaki originally saw the sufficiency direction of his Theorem, although Justin clearly did.

- Other people, like Huffman (1976) and Husimi (1979) discovered the degree 4 case (only) of Kawasaki.

- However, in a 1977 paper, Stewart A. Robertson (Univ. of Southampton, UK) discovered and proved the full necessary direction of Kawasaki, and more!

Proceedings of the Royal Society of Edinburgh, 275–284, 1977

Isometric folding of Riemannian manifolds

S. A. Robertson
Department of Mathematics, University of Southampton

(*Communicated by A. Jeffrey*)

(MS received 28 March 1977. Read 31 October 1977)

SYNOPSIS

When a sheet of paper is crumpled in the hands and then crushed flat against a desk-top, the pattern of creases so formed is governed by certain simple rules. These rules generalize to theorems on folding Riemannian manifolds isometrically into one another. The most interesting results apply to the case in which domain and codomain have the same dimension. The main technique of proof combines the notion of volume with Hopf's concept of the degree of a map.

The ideas in this paper are abstracted from a study of the following familiar actions. Suppose that a plane sheet of paper is crumpled gently in the hands, and then is crushed flat against a desk top. The effect is to criss-cross the sheet with a pattern of creases, which persist even when the sheet is unfolded and smoothed out again to its original planar form. At first sight, the pattern may seem random and chaotic. However, a closer inspection will lead to the following observations.

First of all, the creases appear to be composed of straight line segments. Secondly, if p is the end-point of such a segment, then the total number of crease-segments that end at p is even. (In fact, this number is usually four.) Thirdly, the sum of alternate angles between creases at each such point p is equal to π.

This physical process can be modelled mathematically as follows. Let us replace both the sheet of paper and the desk-top by the Euclidean plane R^2, equipped with its standard flat Riemannian tensorfield. We model the crumpling and crushing process by a map $f: R^2 \to R^2$ that sends each piecewise-straight path in R^2 to a piecewise-straight path of the same length.

More generally, consider two C^∞ Riemannian manifolds M and N, of dimensions m and n respectively. Then a map $f: M \to N$ is said to be an *isometric*

COROLLARY 3. *If $f \in \mathcal{F}(M, N)$ and $\deg f = 0$, then $V_+ = V_- = \frac{1}{2}V \geqq V_f$.*

COROLLARY 4. *If $f \in \mathcal{F}(M, N)$ and f is not surjective, then $\deg f = 0$*

COROLLARY 5. *If $f \in \mathcal{F}(M, M)$, then $\deg f = 0$ or $\deg f = \pm 1$ according as $\sum(f) \neq \varnothing$ or $\sum(\varnothing) = \varnothing$.*

Proof. We can assume without loss of generality that $V = 1$. Hence $V_+ = \alpha$, where $0 \leqq \alpha \leqq 1$ and $V_- = 1 - \alpha$. Let $\deg f = \kappa$. Then

$$\textstyle\sum(f) \neq \varnothing \Leftrightarrow 0 < \alpha < 1 \Leftrightarrow -1 < \kappa V_f < 1 \Leftrightarrow -1 < \kappa < 1 \Leftrightarrow \kappa = 0,$$

and

$$\begin{aligned} \textstyle\sum(f) = \varnothing &\Leftrightarrow 0 = \alpha \quad \text{or} \quad 1 = \alpha \quad \text{and} \quad V_f = 1 \\ &\Leftrightarrow \kappa = 2\alpha - 1 = -1 \quad \text{or} \quad 1. \end{aligned}$$

COROLLARY 6. *Let $f \in \mathcal{F}(M, N)$. Then for all $x \in M$, $f_x : S(x, \mu(x)) \to S(f(x), \mu(x))$ has degree 0 if $x \in \sum(f)$ and had degree ± 1 otherwise.*

Proof. This is essentially a special case of Corollary 5, although $S(x, \mu(x))$ and $S(f(x), \mu(x))$ are distinct manifolds.

3. SURFACES

The results of §2 take a particularly simple form in case M is a surface ($q = 2$). Putting Corollary 2 of §1 together with Theorem 4 and its corollaries, we obtain the following theorem for isometric folding of surfaces.

THEOREM 5. *Let $f \in \mathcal{F}(M, N)$, where both M and N are smooth Riemannian 2-manifolds. Then for each $x \in \sum(f)$, the singularities of f near x form the images of an even number $2r$ of geodesic rays emanating from x, making alternate angles*

$$\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$$

where

$$\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_i = \pi.$$

We call the number r the *order* of the singularity x.

The set of singularities $\sum(f)$ of an isometric folding of a smooth Riemannian surface M into another N is therefore a graph on M, satisfying the angle conditions of Theorem 5. Moreover, in case M is compact and both M and N are oriented, the set $\sum(f)$ must partition M in accordance with the area conditions of Theorem 4. Note that, by Theorem 5, every vertex of the graph $\sum(f)$ has even valency. Of course $\sum(f)$ need not be connected, and may have components homeomorphic to a circle and having no vertices.

Figure 5 shows the positive (shaded) and negative (unshaded) 2-strata into which a double torus S could be partitioned by the singularities of an isometric folding f of S into itself. The image of f in this example is a 'quarter' of S, homeomorphic to a cylinder $S^1 \times I$.

3. Surfaces

The results of §2 take a particularly simple form in case M is a surface $(q=2)$. Putting Corollary 2 of §1 together with Theorem 4 and its corollaries, we obtain the following theorem for isometric folding of surfaces.

THEOREM 5. *Let $f \in \mathcal{F}(M, N)$, where both M and N are smooth Riemannian 2-manifolds. Then for each $x \in \sum(f)$, the singularities of f near x form the images of an even number $2r$ of geodesic rays emanating from x, making alternate angles*

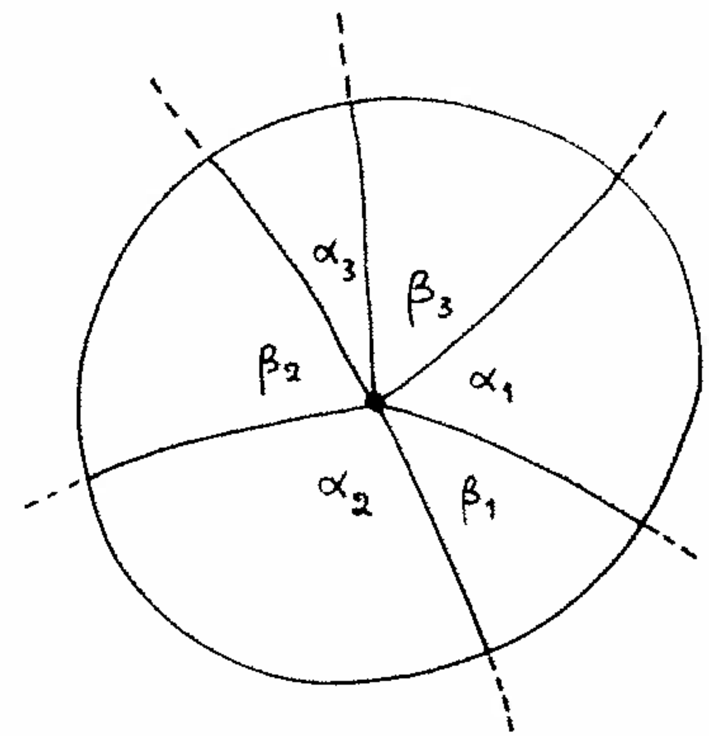
$$\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$$

where

$$\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_i = \pi.$$

We call the number r the *order* of the singularity x.

The set of singularities $\sum(f)$ of an isometric foldir



$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = \pi$

Figure 4. A singularity of order 3.

So what does Robertson prove?

- Let M be a smooth (C^∞) compact, oriented n-manifold without boundary.

So what does Robertson prove?

- Let M be a smooth (C^∞) compact, oriented n-manifold without boundary.
- Let $f : M \to M$ be an isometric folding: If γ is any piecewise geodesic curve on M parameterized with respect to arc length, then so is $f(\gamma)$.

So what does Robertson prove?

- Let M be a smooth (C^∞) compact, oriented n-manifold without boundary.

- Let $f : M \to M$ be an isometric folding: If γ is any piecewise geodesic curve on M parameterized with respect to arc length, then so is $f(\gamma)$.

- This implies that f is continuous, but not necessarily differentiable.
  Let $\Sigma(f)$ be the set of all singularities of f. (This is the crease pattern.)

So what does Robertson prove?

- Let M be a smooth (C^∞) compact, oriented n-manifold without boundary.
- Let $f : M \to M$ be an isometric folding: If γ is any piecewise geodesic curve on M parameterized with respect to arc length, then so is $f(\gamma)$.
- This implies that f is continuous, but not necessarily differentiable. Let $\Sigma(f)$ be the set of all singularities of f. (This is the crease pattern.)
- Robertson then proves that $\Sigma(f)$ forms an n-dimensional cell-complex, which is the union of *strata* that are n-1, n-2, n-3, ..., 3, 2, 1-dimensional. (The n-dim cells are also *strata*, but they're not part of $\Sigma(f)$.) Also, the number of such *strata* is finite.

So what does Robertson prove?

- Each of the *strata* of $\Sigma(f)$ is isometrically immersed in $f(M)$ as a geodesic submanifold of M.

- Thus for almost all $y \in f(M)$ we have $f^{-1}(y) = \{x_1, ..., x_v\}$ where each $x_i \in M - \Sigma(f)$.

So what does Robertson prove?

- Each of the *strata* of $\Sigma(f)$ is isometrically immersed in $f(M)$ as a geodesic submanifold of M.
- Thus for almost all $y \in f(M)$ we have $f^{-1}(y) = \{x_1, ..., x_v\}$ where each $x_i \in M - \Sigma(f)$.
- For each n-dimensional *stratum* S of $f(M)$, call S *positive* or *negative* depending on whether f is orientation-preserving or not on S.

So what does Robertson prove?

- Each of the *strata* of $\Sigma(f)$ is isometrically immersed in $f(M)$ as a geodesic submanifold of M.

- Thus for almost all $y \in f(M)$ we have $f^{-1}(y) = \{x_1, ..., x_v\}$ where each $x_i \in M - \Sigma(f)$.

- For each n-dimensional *stratum* S of $f(M)$, call S *positive* or *negative* depending on whether f is orientation-preserving or not on S.

- If λ points of $f^{-1}(y)$ are positive and μ are negative, then we have that $\deg f = \lambda - \mu$.
  (The degree of a map is, intuitively, the number of times f wraps M around $f(M)$.)

So what does Robertson prove?

- Now, still working with an n-manifold M, we define:
  V = the n-dimensional volume of M.
  V_+ = the n-volume of the positive n-dim *strata*.
  V_- = the n-volume of the negative n-dim *strata*.
  V_f = the n-volume of $f(M)$. (So $V = V_+ + V_-$.)

So what does Robertson prove?

- Now, still working with an n-manifold M, we define:
  V = the n-dimensional volume of M.
  V_+ = the n-volume of the positive n-dim *strata*.
  V_- = the n-volume of the negative n-dim *strata*.
  V_f = the n-volume of $f(M)$. (So $V = V_+ + V_-$.)

- **Robertson's Theorem:** Let $f : M \to M$ be an isometric folding and let $\deg f = k$.

  Then $V_+ = V_- + kV_f$.

So what does Robertson prove?

- Now, still working with an n-manifold M, we define:
  V = the n-dimensional volume of M.
  V_+ = the n-volume of the positive n-dim *strata*.
  V_- = the n-volume of the negative n-dim *strata*.
  V_f = the n-volume of $f(M)$. (So $V = V_+ + V_-$.)

- **Robertson's Theorem:** Let $f : M \to M$ be an isometric folding and let $\deg f = k$.
  Then $V_+ = V_- + kV_f$.

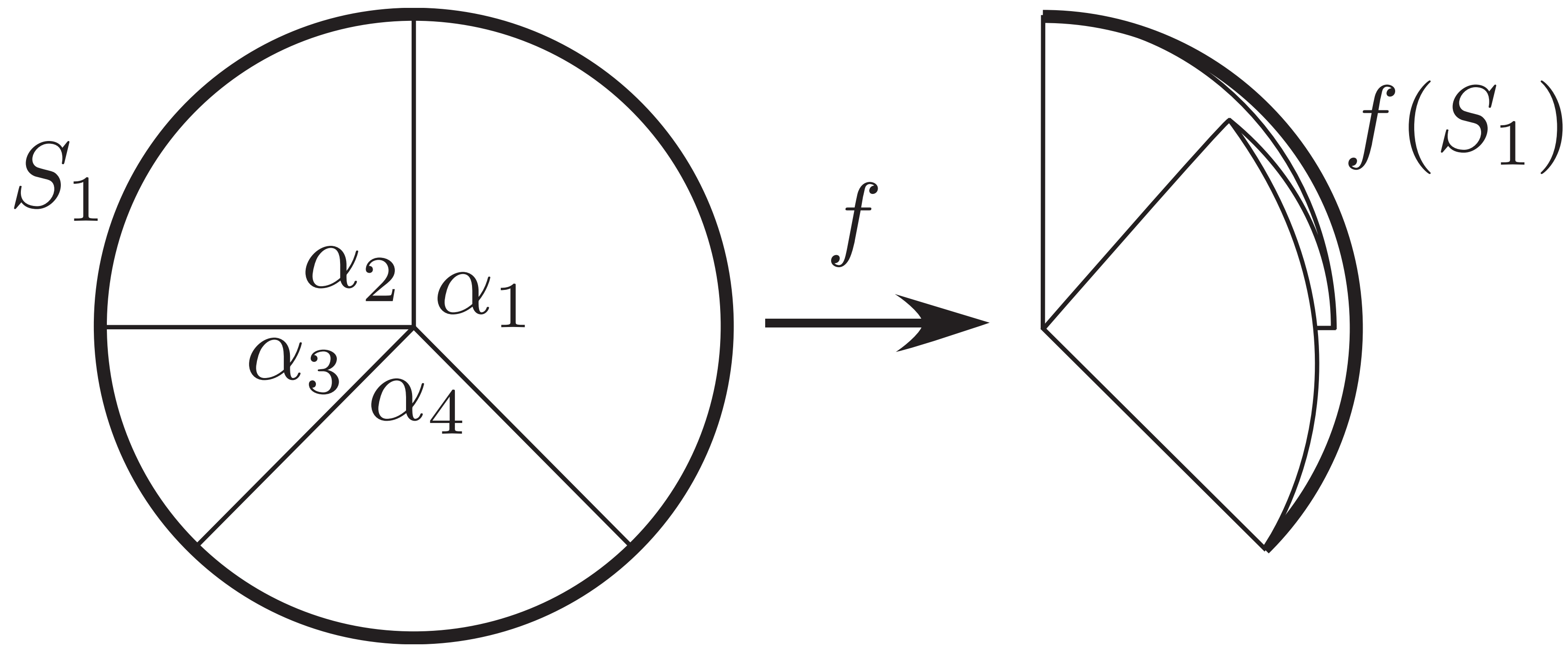
  Proof: Since kV_f counts the volume of V_f, summing all the (signed) layers in $f(M)$, we have

  $$kV_f = V_+ - V_-$$

  which gives the desired result.

So what?

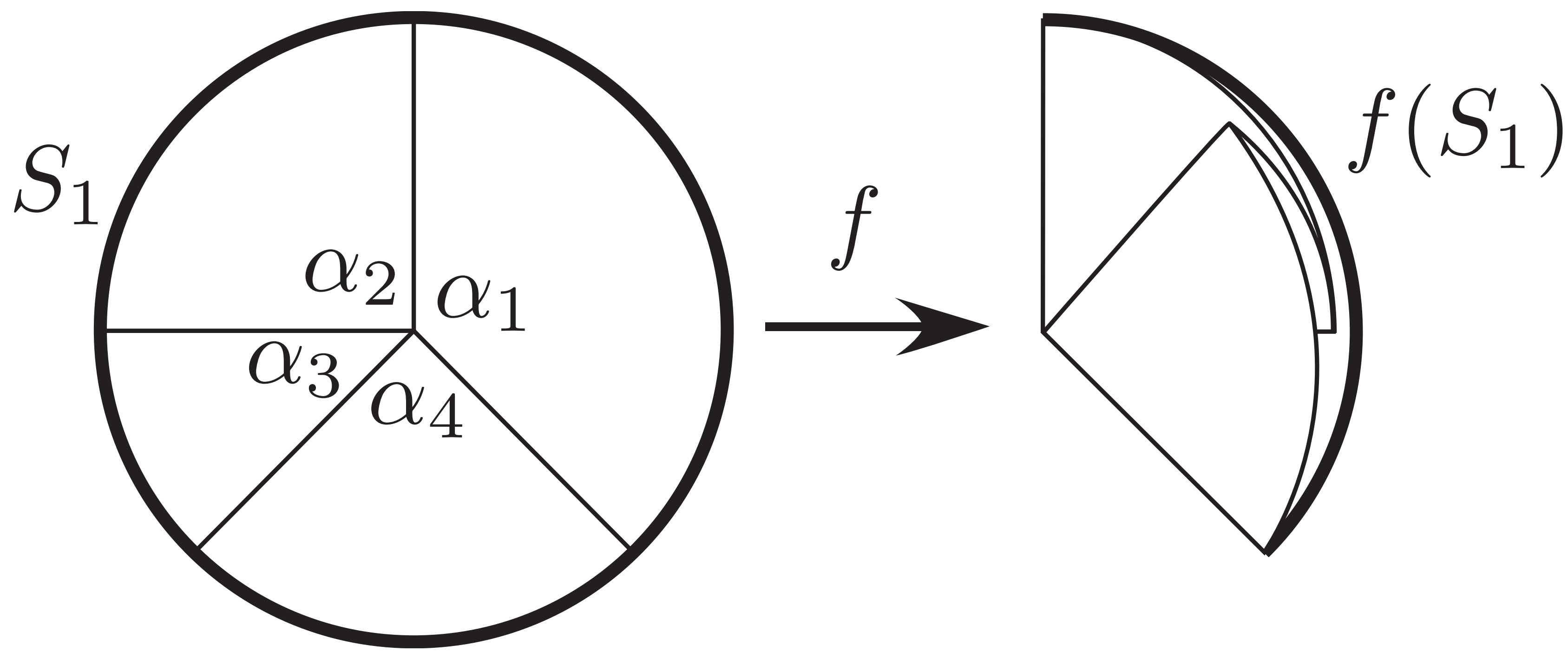
- Take a vertex in a flat origami crease pattern, and draw a circle of radius 1 around it (rescaling your c.p. if necessary).



The boundary of your circle is the 1-manifold S_1, and it folds isometrically into S_1.

So what?

- Take a vertex in a flat origami crease pattern, and draw a circle of radius 1 around it (rescaling your c.p. if necessary).



The boundary of your circle is the 1-manifold S_1, and it folds isometrically into S_1.

- The 1-volumes V_+ will be the angles α_i for i odd, and V_- for i even.
- Also, $\deg f = 0$ here, so $V_+ = V_- + kV_f$ becomes

$$\alpha_1 + \alpha_3 + \cdots \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}$$

Wait, how to we know that $\deg f = 0$?

- **Corollary to Robertson:** If f is an isometric folding (in any dimension) that is not surjective (onto), then $\deg f = 0$.

Wait, how to we know that $\deg f = 0$?

- **Corollary to Robertson:** If f is an isometric folding (in any dimension) that is not surjective (onto), then $\deg f = 0$.

- **Proof:** If f is not surjective, then $f(M)$ has boundary in the range. For any point $y \in f(M)$ near such boundary, the preimage $f^{-1}(y)$ will contain an even number of points, half positive and half negative. Thus

$$\deg f = \lambda - \mu = 0$$

Wait, how to we know that $\deg f = 0$?

- **Corollary 2 to Robertson:** If $f : M \to M$ is an isometric folding, then $\deg f = 0$ iff $\Sigma(f) \neq \emptyset$ and $\deg f = \pm 1$ iff $\Sigma(f) = \emptyset$.

Wait, how to we know that $\deg f = 0$?

- **Corollary 2 to Robertson:** If $f : M \to M$ is an isometric folding, then $\deg f = 0$ iff $\Sigma(f) \neq \emptyset$ and $\deg f = \pm 1$ iff $\Sigma(f) = \emptyset$.

- **Proof:** Assume $V = 1$. Let $V_+ = a$ where $0 \leq a \leq 1$ and $V_- = 1 - a$ Let $\deg f = k$.

Wait, how to we know that $\deg f = 0$?

- **Corollary 2 to Robertson:** If $f : M \to M$ is an isometric folding, then $\deg f = 0$ iff $\Sigma(f) \neq \emptyset$ and $\deg f = \pm 1$ iff $\Sigma(f) = \emptyset$.

- **Proof:** Assume $V = 1$. Let $V_+ = a$ where $0 \leq a \leq 1$ and $V_- = 1 - a$ Let $\deg f = k$.

  If $\Sigma(f) \neq \emptyset$ then f is not surjective, so $\deg f = 0$ by Corollary 1.

  And if $\deg f = 0$ then
  $$V_+ - V_- = kV_f = 0 \Rightarrow V_+ = V_- = \frac{1}{2}V \geq V_f$$
  so f is surjective.

Wait, how to we know that $\deg f = 0$?

- **Corollary 2 to Robertson:** If $f : M \to M$ is an isometric folding, then $\deg f = 0$ iff $\Sigma(f) \neq \emptyset$ and $\deg f = \pm 1$ iff $\Sigma(f) = \emptyset$.

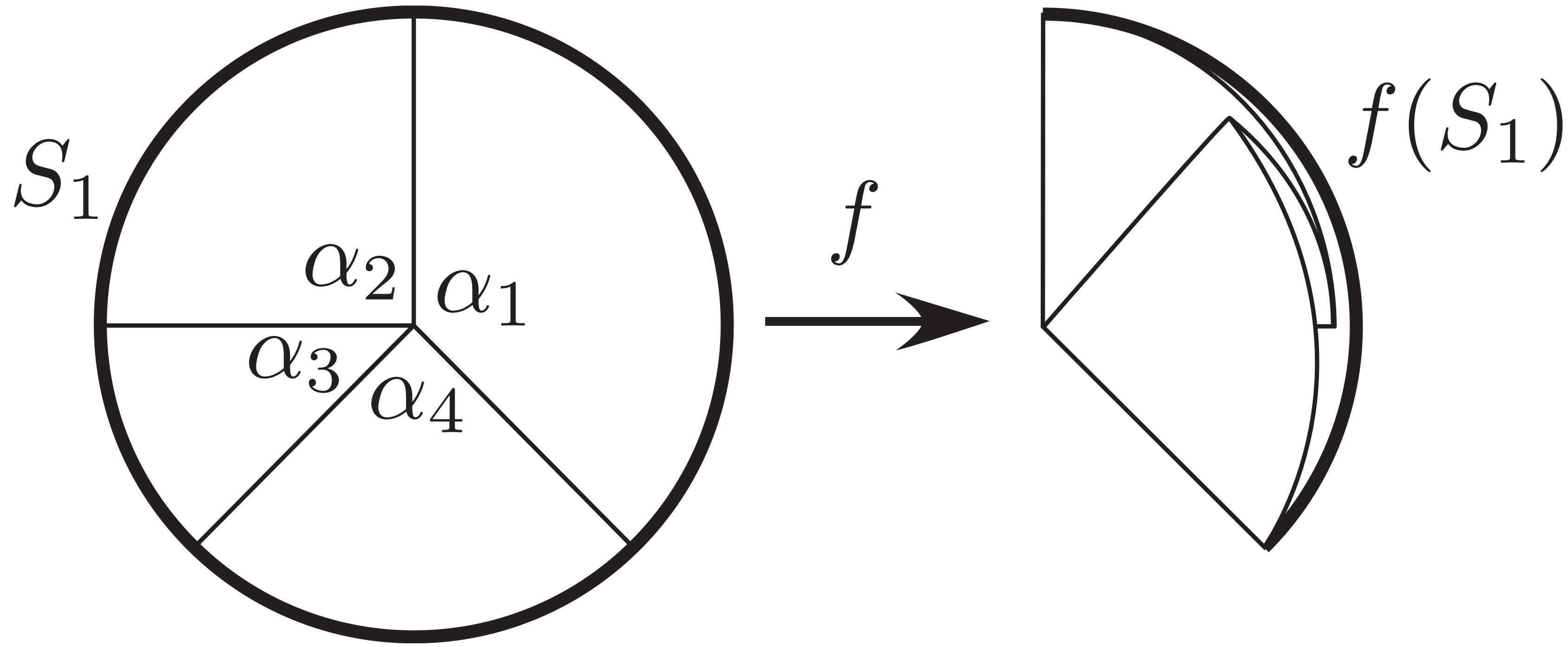
- **Proof:** Assume $V = 1$. Let $V_+ = a$ where $0 \leq a \leq 1$ and $V_- = 1 - a$ Let $\deg f = k$.

  On the other hand,

$$\begin{aligned} \Sigma(f) = \emptyset &\Leftrightarrow (a = 0 \text{ or } a = 1) \text{ and } V_f = 1 \\ &\Leftrightarrow k = V_+ - V_- = 2a - 1 = \pm 1 \end{aligned}$$

So Robertson proved the $\Rightarrow$ direction of Kawasaki's Theorem

- Take a vertex in a flat origami crease pattern, and draw a circle of radius 1 around it (rescaling your c.p. if necessary).



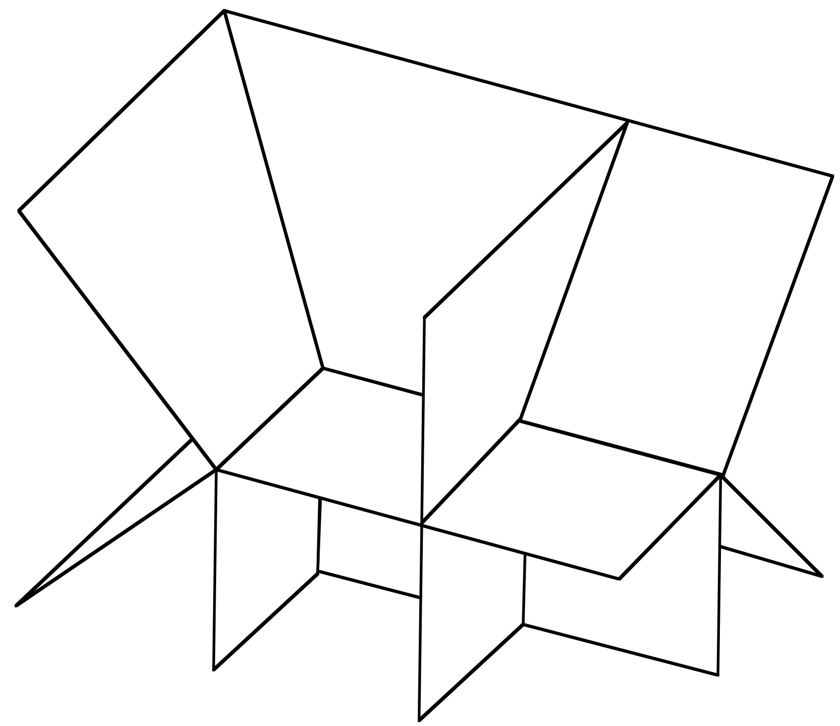
  The boundary of your circle is the 1-manifold S_1, and it folds isometrically into S_1.

- The 1-volumes V_+ will be the angles α_i for i odd, and V_- for i even.
- Also, $\deg f = 0$ here, so $V_+ = V_- + kV_f$ becomes

$$\alpha_1 + \alpha_3 + \cdots \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}$$

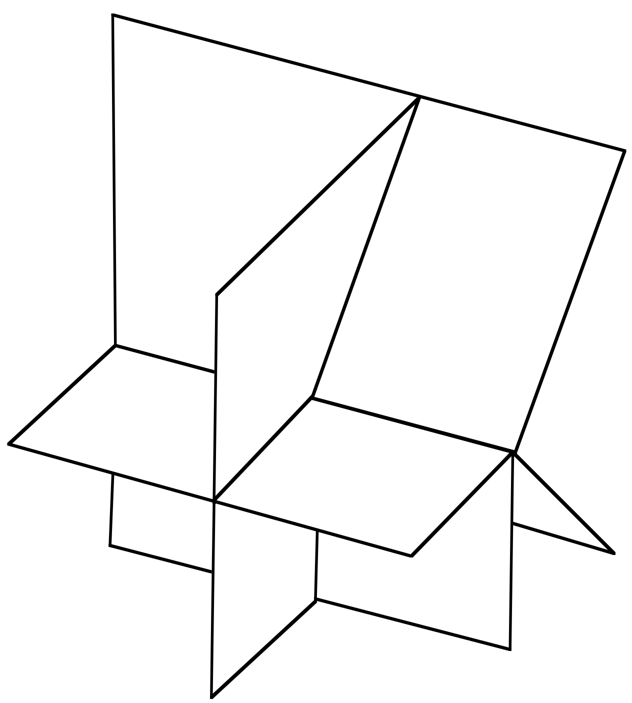
What does Robertson’s Theorem say about other dimensions?

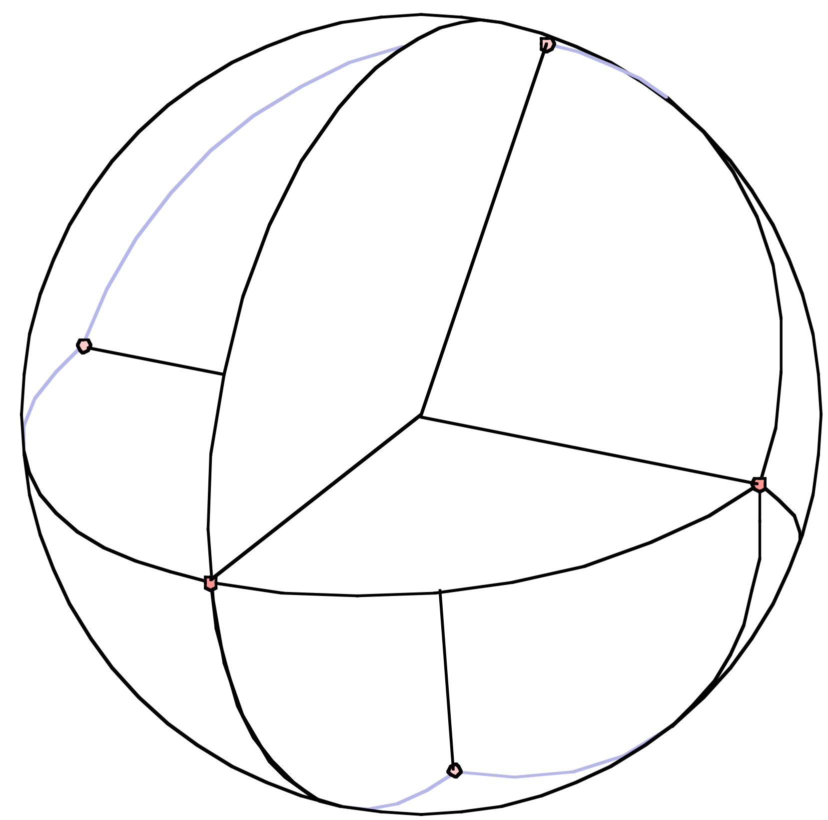
- Suppose we “fold” a chunk of 3D space. Our “crease lines” are parts of planes, and “folding” means reflecting space through those planes.



What does Robertson’s Theorem say about other dimensions?

- Suppose we “fold” a chunk of 3D space. Our “crease lines” are parts of planes, and “folding” means reflecting space through those planes.





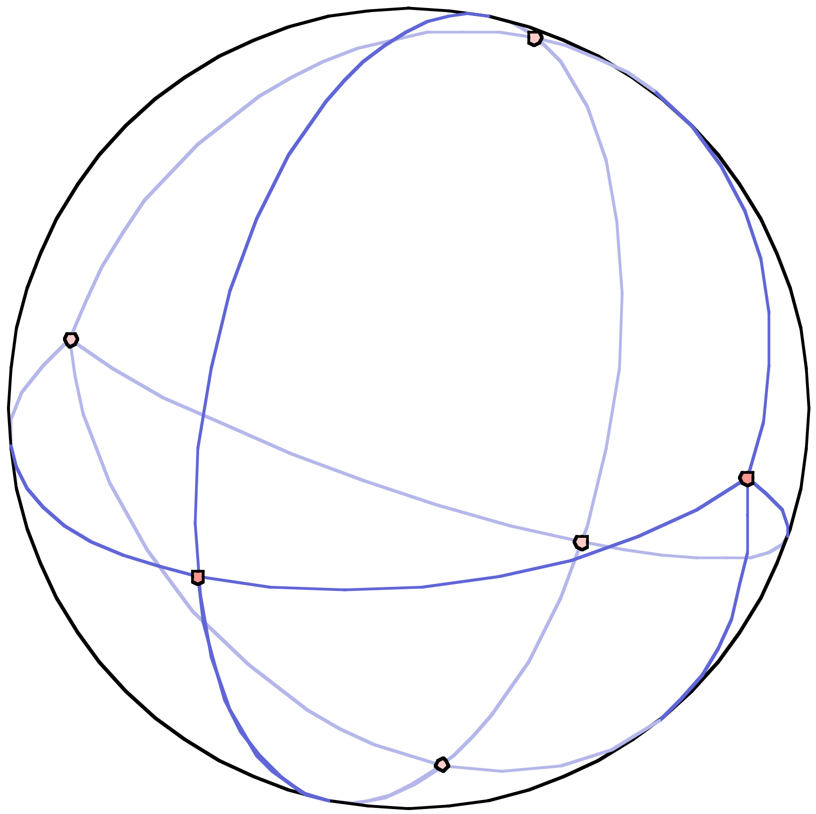
- Take one vertex in such a crease pattern and draw a sphere of radius 1 around it. Where the planes intersect this sphere will create a crease pattern on the 2-manifold S_2.

What does Robertson's Theorem say about other dimensions?

- Suppose we "fold" a chunk of 3D space. Our "crease lines" are parts of planes, and "folding" means reflecting space through those planes.
- If our original 3D vertex "folds flat" then our crease pattern on S_2 is an isometric folding $f : S_2 \to S_2$.
- Robertson's Theorem says that

$$V_+ - V_- = kV_f = 0$$

And here the volumes are the areas of the spherical polygons.



What does Robertson's Theorem say about other dimensions?

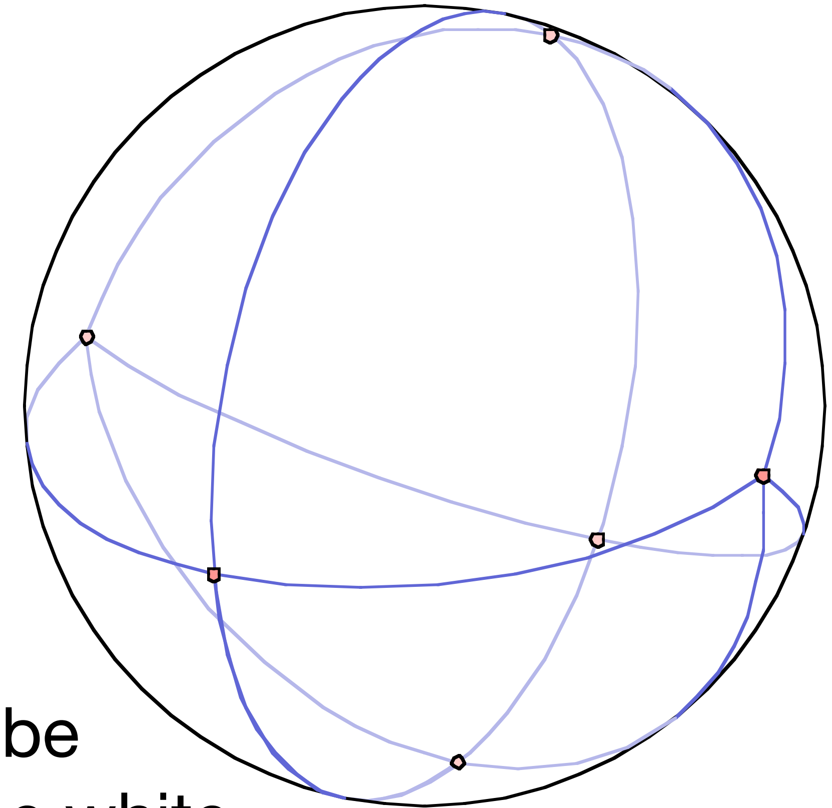
- Suppose we "fold" a chunk of 3D space. Our "crease lines" are parts of planes, and "folding" means reflecting space through those planes.

- If our original 3D vertex "folds flat" then our crease pattern on S_2 is an isometric folding $f : S_2 \to S_2$.

- Robertson's Theorem says that

$$V_+ - V_- = kV_f = 0$$

  And here the volumes are the areas of the spherical polygons.

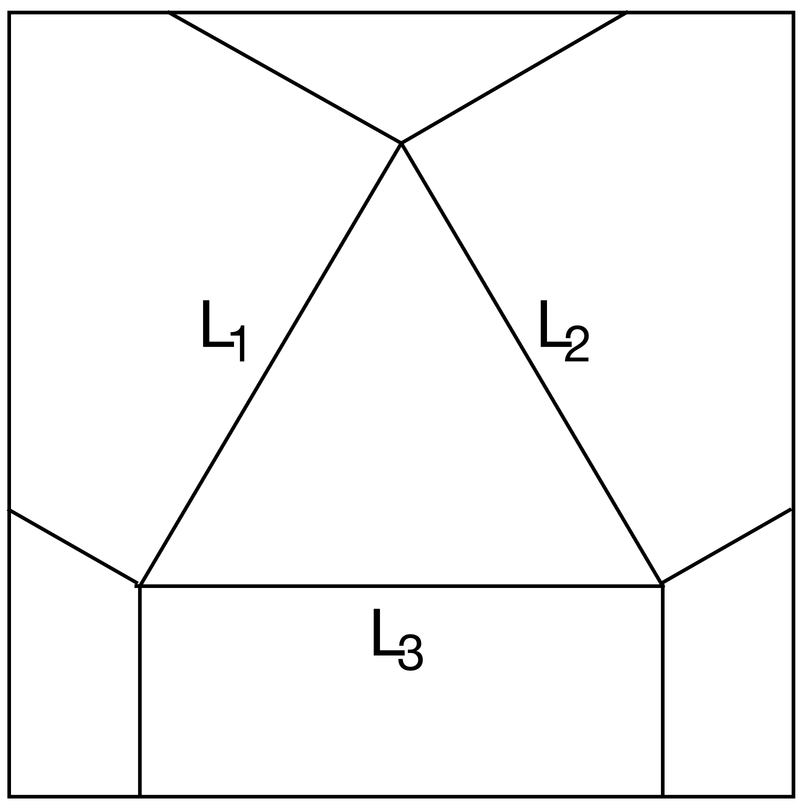


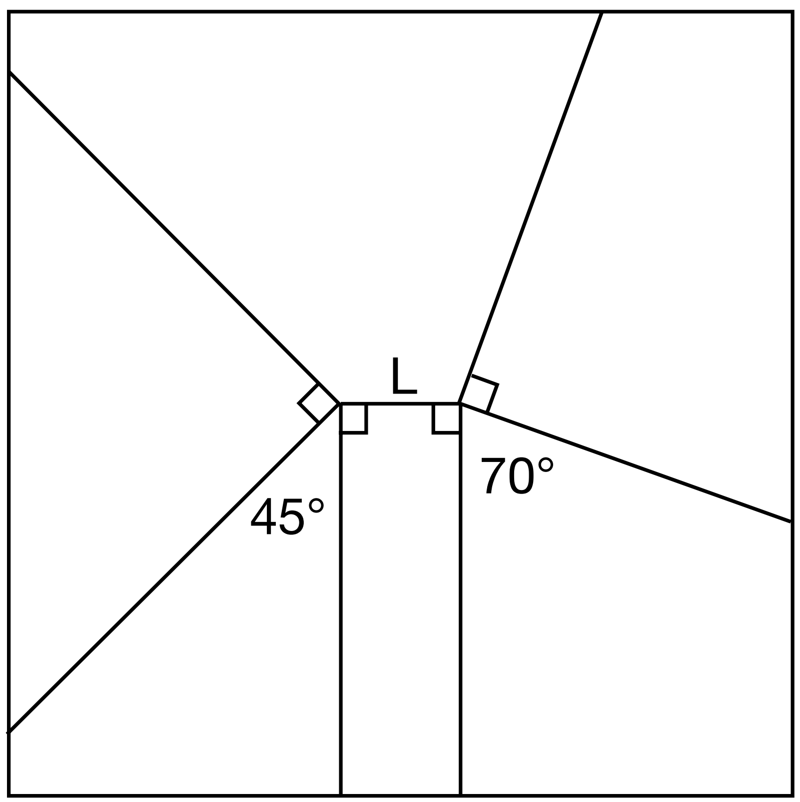
- So 2-color these regions, and let $B_1, ..., B_k$ be the black region areas and $W_1, ..., W_k$ be the white. Then

$$\sum B_i = \sum W_i = 2\pi$$

Generalizing ... can cause problems

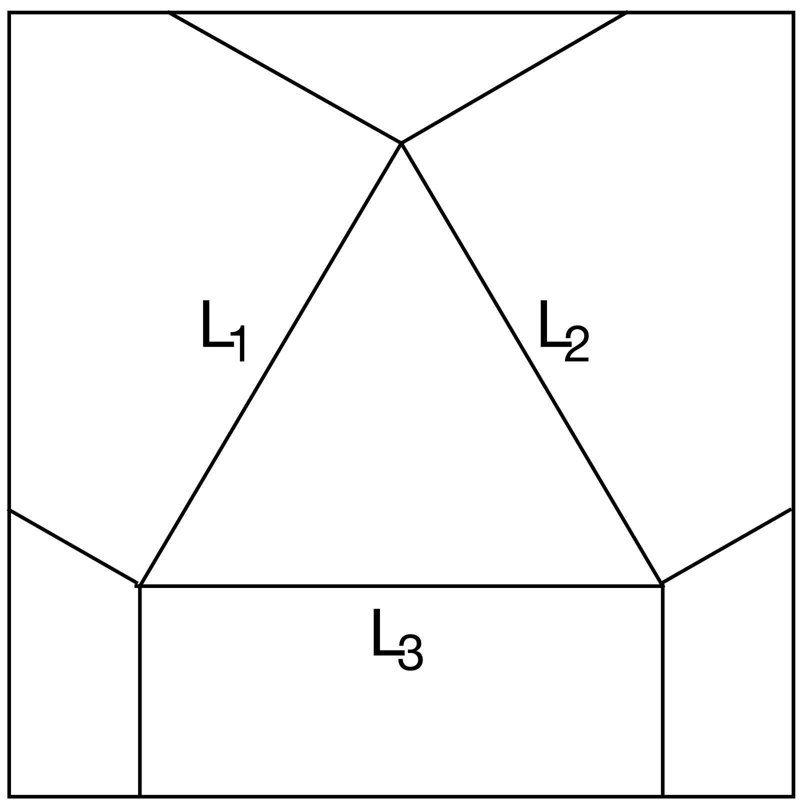
- Kawasaki's Theorem (sufficiency part) does not generalize to larger crease patterns.

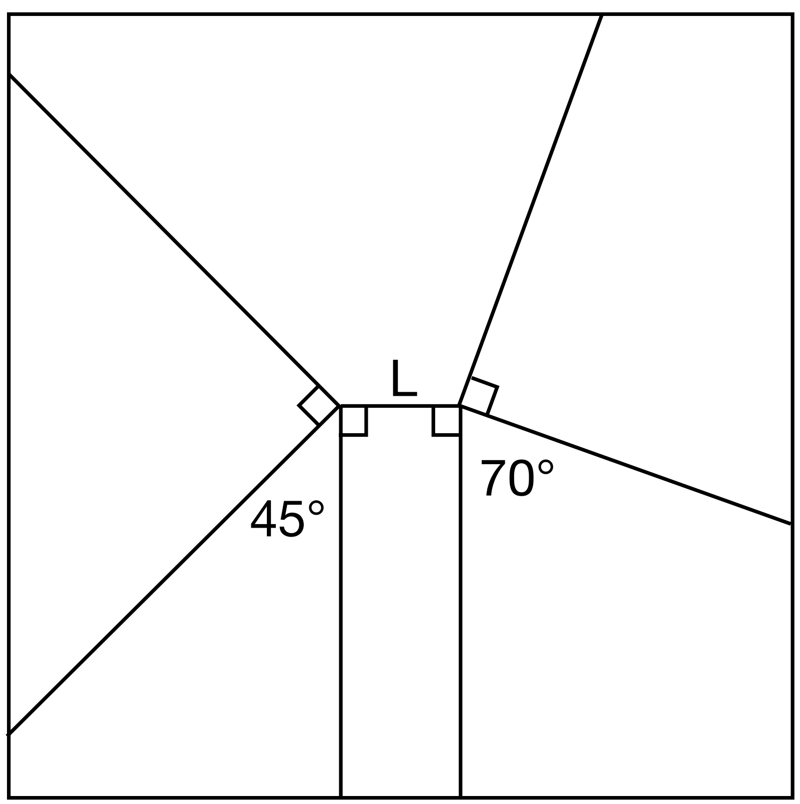




Generalizing ... can cause problems

- Kawasaki's Theorem (sufficiency part) does not generalize to larger crease patterns.





Determining if a given crease pattern is flat-foldable is NP-hard (Bern & Hayes, 1996)

Generalizing ... can be cool

- **Justin's Theorem:** Given any flat origami model, let R be a simple, closed, vertex-avoiding curve drawn on the crease pattern that crosses creases c_1, c_2, c_3, ..., c_{2n}, in order. Let $\alpha_1, \alpha_2, ..., \alpha_{2n}$ be the angles between these crease lines (determined consistently), and let M and V be the number of mountain and valley creases among c_1, ..., c_{2n}. Then

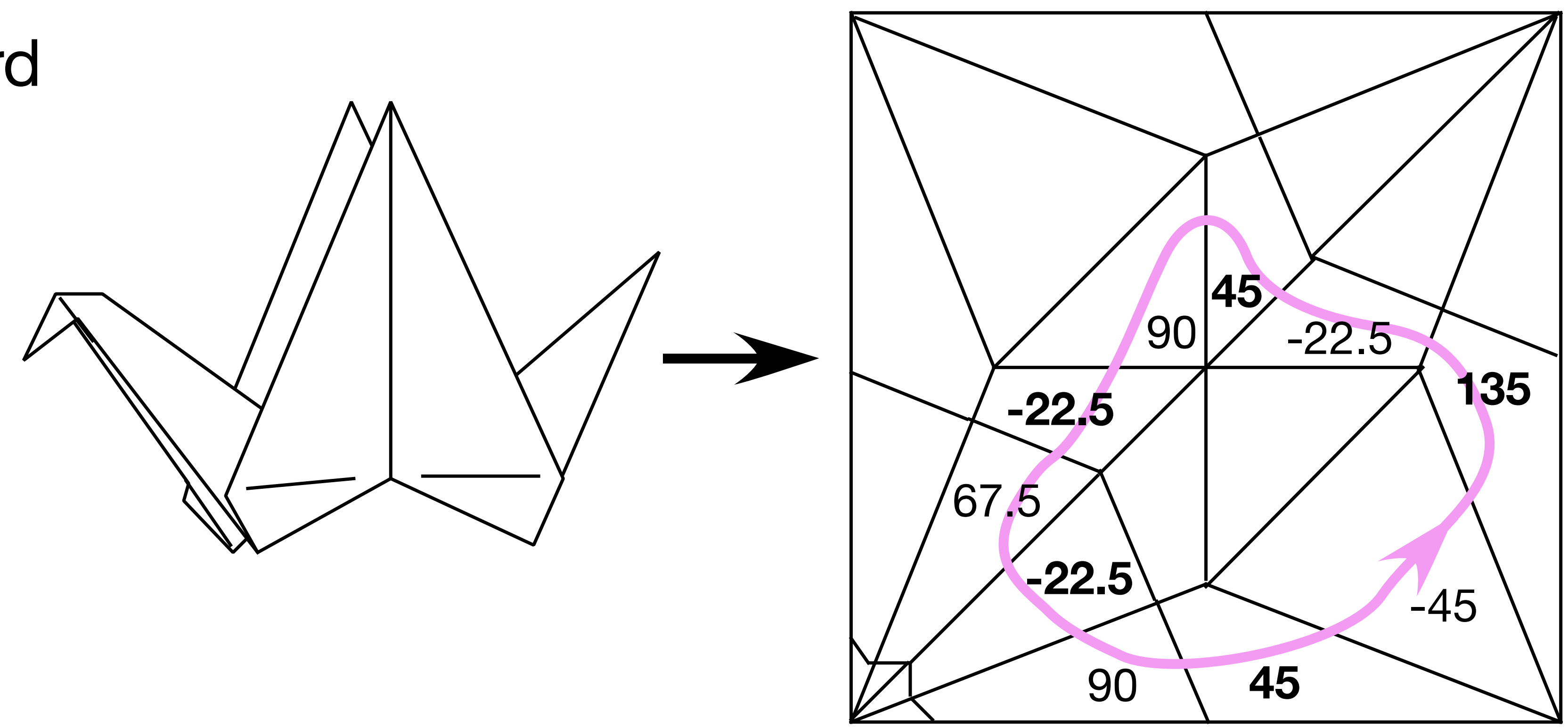
$$\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \frac{M-V}{2}\pi \mod 2\pi$$

Generalizing ... can be cool

- **Justin's Theorem:** Given any flat origami model, let R be a simple, closed, vertex-avoiding curve drawn on the crease pattern that crosses creases c_1, c_2, c_3, ..., c_{2n}, in order. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ be the angles between these crease lines (determined consistently), and let M and V be the number of mountain and valley creases among c_1, ..., c_{2n}. Then

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \frac{M-V}{2}\pi \mod 2\pi$$

- Example: The Flapping Bird



Generalizing ... can be cool

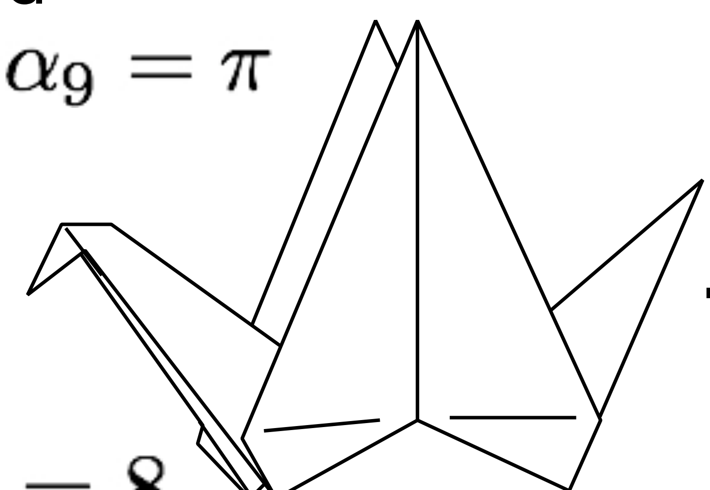
- **Justin's Theorem:** Given any flat origami model, let R be a simple, closed, vertex-avoiding curve drawn on the crease pattern that crosses creases c_1, c_2, c_3, ..., c_{2n}, in order. Let $\alpha_1, \alpha_2, ..., \alpha_{2n}$ be the angles between these crease lines (determined consistently), and let M and V be the number of mountain and valley creases among c_1, ..., c_{2n}. Then

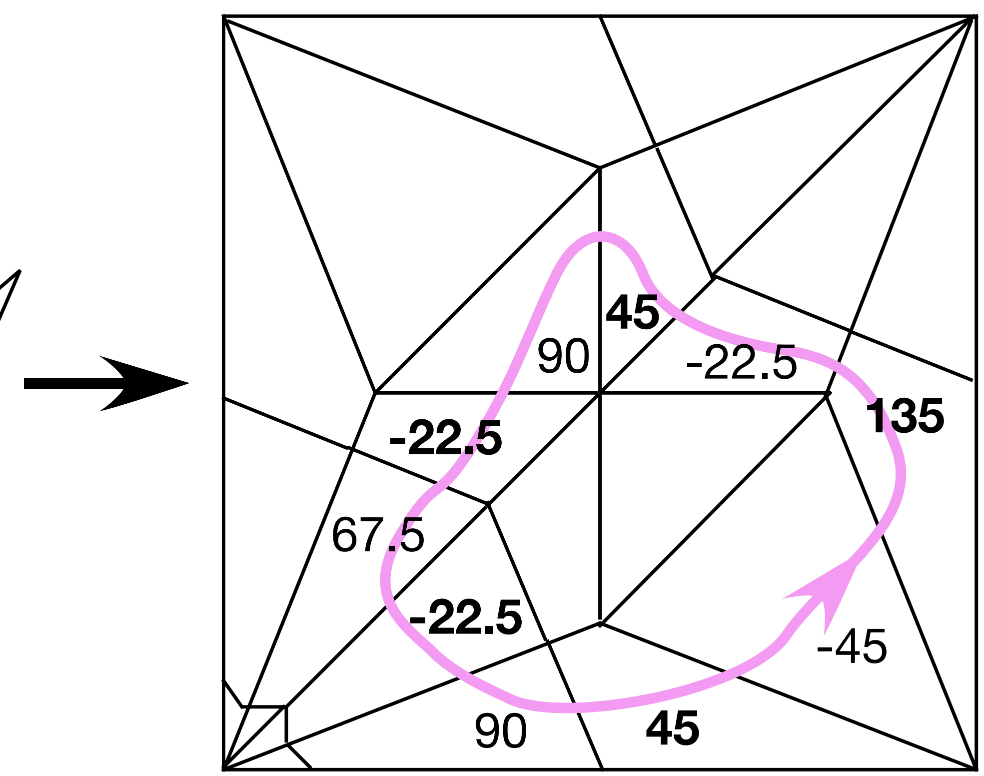
$$\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \frac{M-V}{2}\pi \mod 2\pi$$

- Example: The Flapping Bird
  Here $\alpha_1 + \alpha_3 + \cdots + \alpha_9 = \pi$
  so Justin says that

$$\frac{M-V}{2} = 1 \mod 2$$

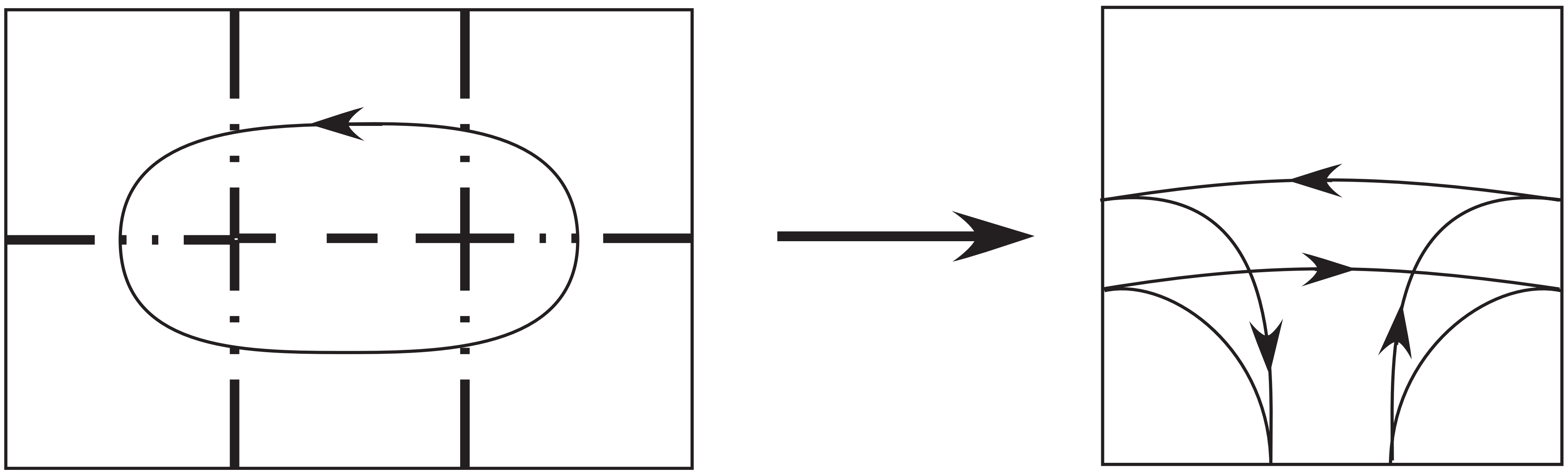
  Checking, we see that $M = 8$ and $V = 2$, so $M - V = 6$ which works!





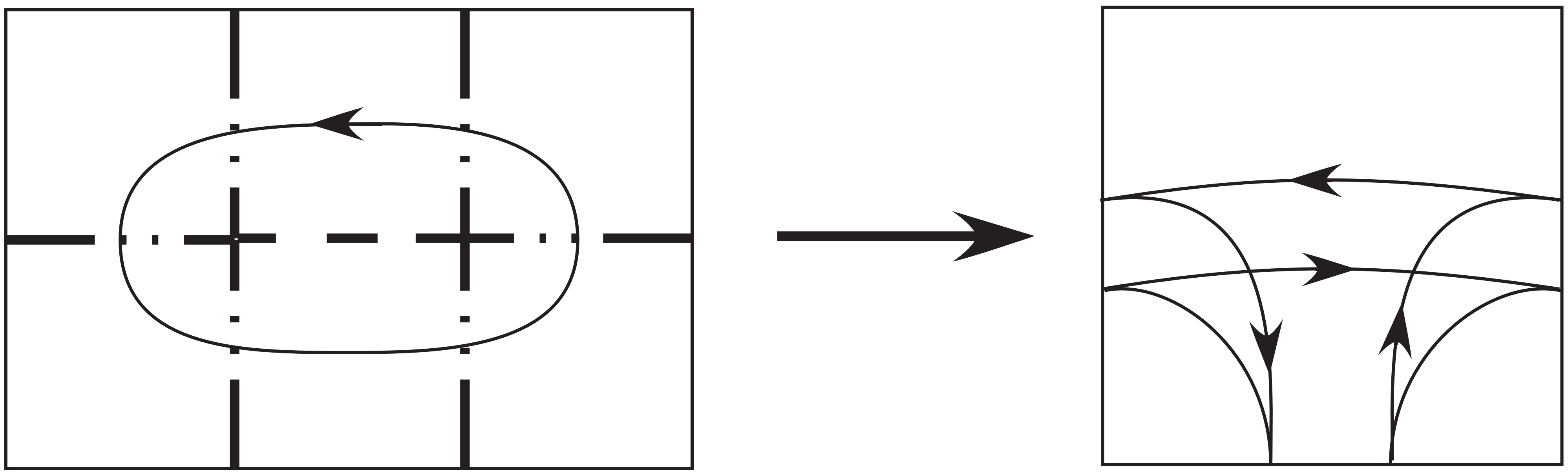
Proving Justin's Theorem

- Look at what happens to our closed curve after we fold the paper.



Proving Justin's Theorem

- Look at what happens to our closed curve after we fold the paper.



- Following the image of the closed curve on our folded paper, it will turn around some number of of times.

  So the turning of our image path $= 0 \mod 2\pi$.

Proving Justin's Theorem

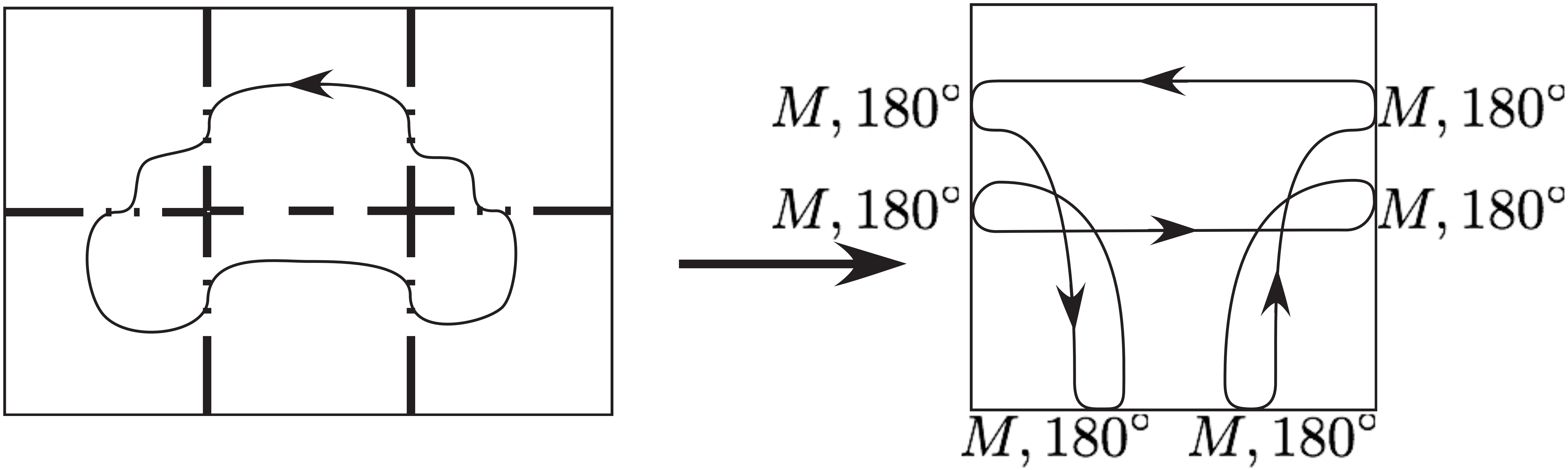
- Let's keep better track of the total turning of the image curve by picking a better curve to begin with.

We approach each crease line perpendicular to it.
If we make our curve cross each crease line while tangent to it (at an inflection point), then after it is folded we will have a 180° turn.

What's more, if we do this right then we can have each M crease turn 180° and each V crease turn -180°!

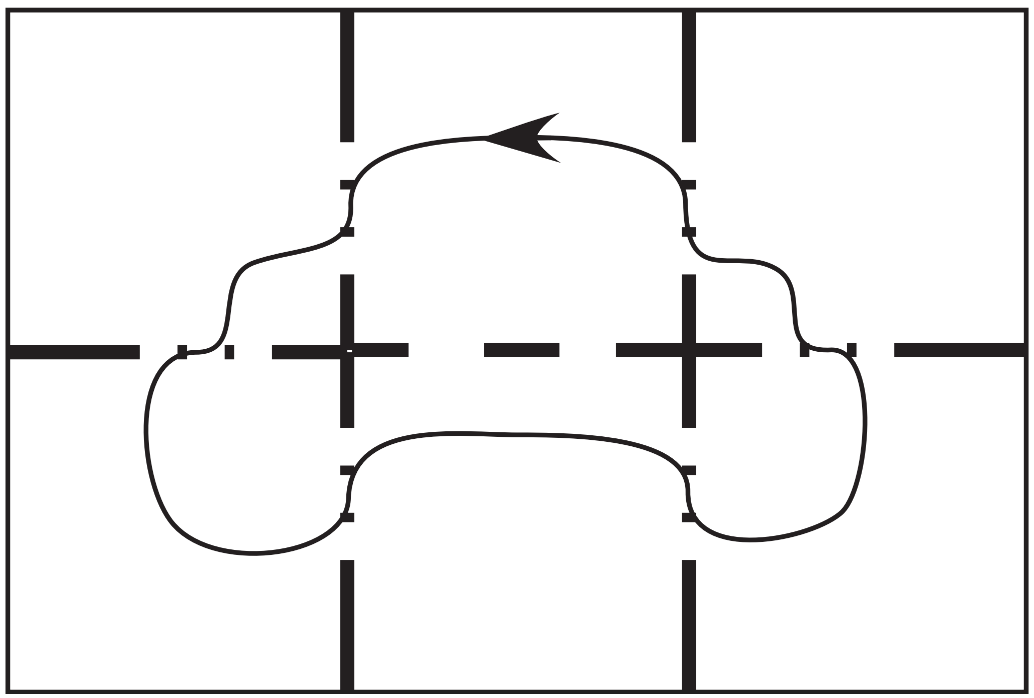
Proving Justin’s Theorem

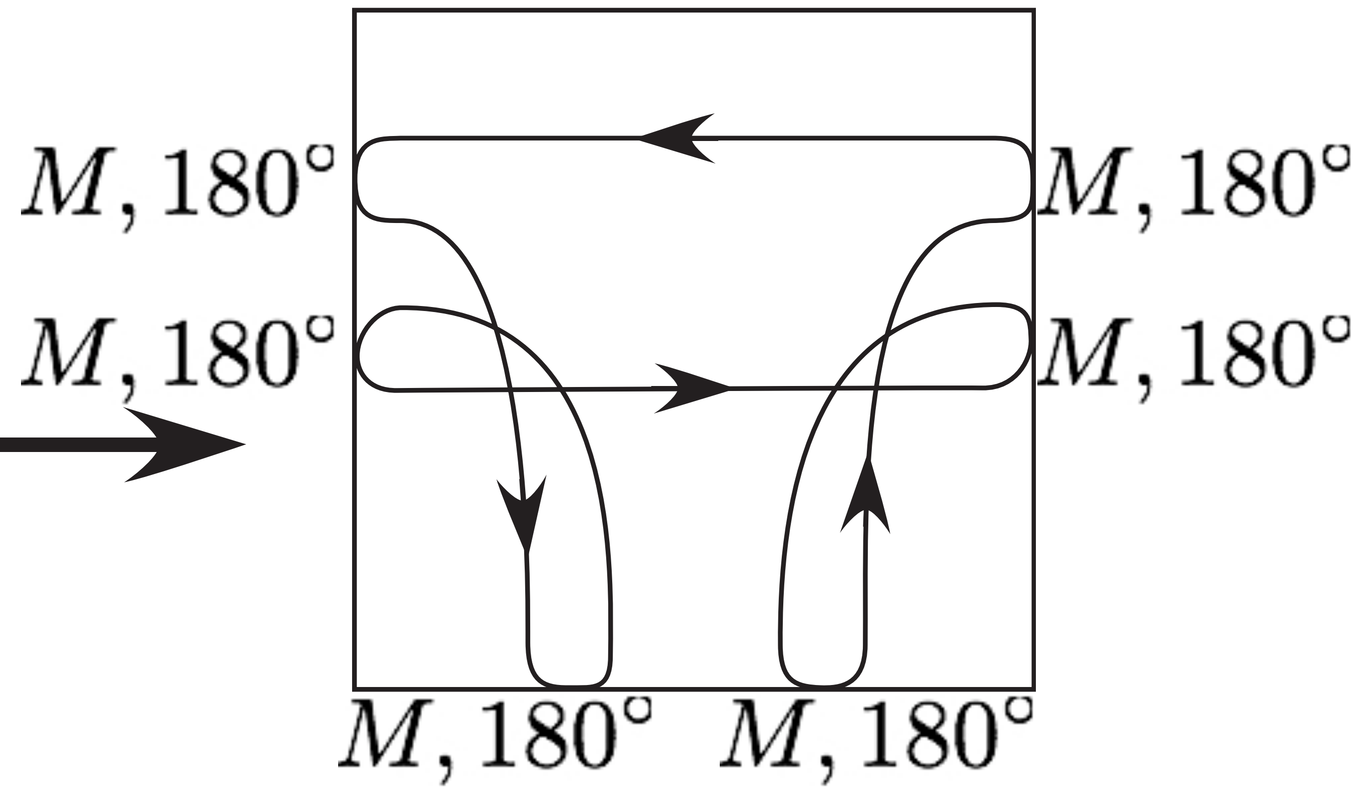
- Some examples:



Proving Justin's Theorem

- Some examples:





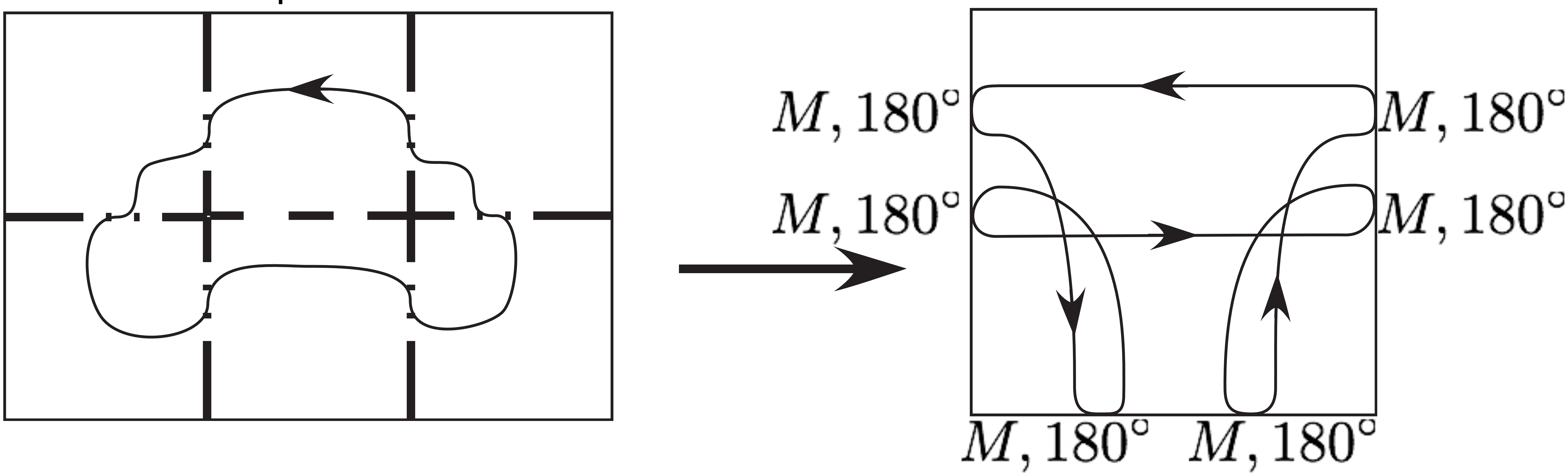
So the total turning of the image curve

$$= \alpha_1 - \alpha_2 + \cdots - \alpha_6 + M\pi - V\pi$$

which should be congruent to $0 \mod 2\pi$.

Proving Justin’s Theorem

- Some examples:



So the total turning of the image curve

$$= \alpha_1 - \alpha_2 + \cdots - \alpha_6 + M\pi - V\pi$$

which should be congruent to $0 \mod 2\pi$.

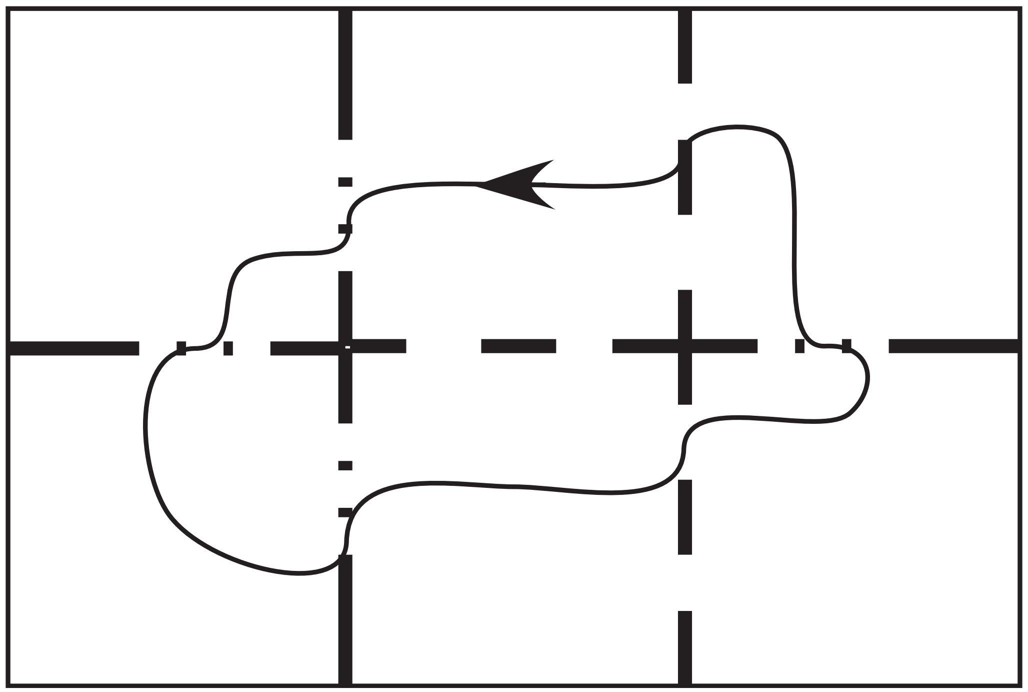
$$= 90^\circ - 90^\circ + 0 - 90^\circ + 90^\circ + 6 \cdot 180^\circ$$

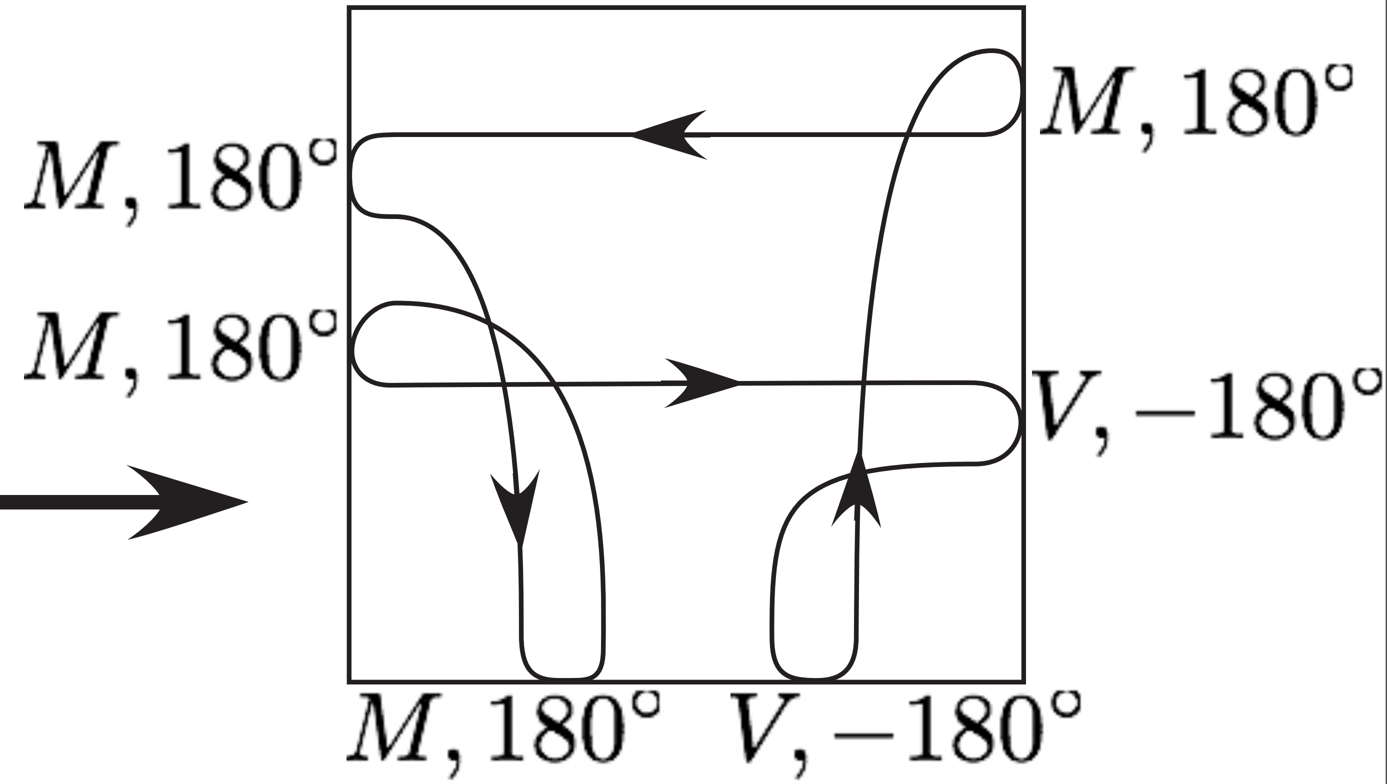
$$= 3 \cdot 2\pi$$

Yes, it is!

Proving Justin's Theorem

- Changing some of the mountains and valleys:





Here the total turning of the image curve is

$$= \alpha_1 - \alpha_2 + \cdots - \alpha_6 + M\pi - V\pi$$

$$= 0 + 4 \cdot \pi - 2 \cdot \pi = 2\pi \equiv 0 \mod 2\pi$$

Proving Justin's Theorem

- Finally, the proof is that we have:

$$\alpha_1 - \alpha_2 + \cdots - \alpha_{2n} + M\pi - V\pi \equiv 0 \mod 2\pi$$

Proving Justin's Theorem

- Finally, the proof is that we have:

$$\alpha_1 - \alpha_2 + \cdots - \alpha_{2n} + M\pi - V\pi \equiv 0 \mod 2\pi$$

add to this the fact that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_{2n} = 0 \mod 2\pi$$

to get

Proving Justin's Theorem

- Finally, the proof is that we have:

$$\alpha_1 - \alpha_2 + \cdots - \alpha_{2n} + M\pi - V\pi \equiv 0 \mod 2\pi$$

add to this the fact that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_{2n} = 0 \mod 2\pi$$

to get

$$2\alpha_1 + 2\alpha_3 + 2\alpha_5 + \cdots + 2\alpha_{2n-1} = (M - V)\pi \mod 2\pi$$

or

$$\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{2n-1} = \frac{(M - V)\pi}{2} \mod 2\pi$$

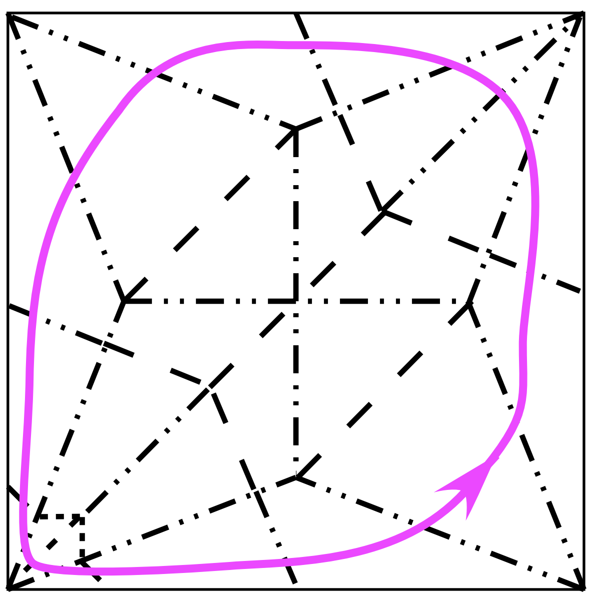
Ditto for the other angles.

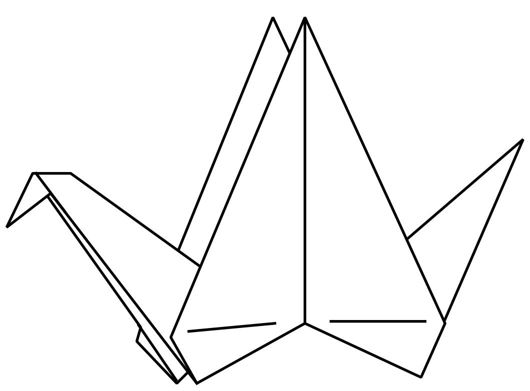
Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.





Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

  For origami, we have $K = 0$ because the paper is flat!

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

  For origami, we have $K = 0$ because the paper is flat!

  Let's let M be the region of paper inside our curve, **after** the paper is folded! (That is, M is homeomorphic to a disc (a disc folded up).)

  Then $\chi(M) = 1$.

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then

$$\int\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$

  where $\chi(M)$ is the Euler characteristic of M.

  So for our manifold we have: $\displaystyle\int_{\partial M} k_g\, ds = 2\pi$

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

  So for our manifold we have: $$\int_{\partial M} k_g\,ds = 2\pi$$

  Now, k_g measures the curvature of ∂M at each point along the curve.

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

  So for our manifold we have: $$\int_{\partial M} k_g\, ds = 2\pi$$

  Now, k_g measures the curvature of ∂M at each point along the curve.

  Between two creases l_i and l_{i+1} we have $k_g = \alpha_i$ or $k_g = -\alpha_i$.

Another way to prove Justin?

- Use the **Gauss-Bonnet Theorem**!
  Let M be a compact 2-D manifold (i.e., surface) with boundary ∂M. Let K be the Gaussian curvature of M and k_g the geodesic curvature of ∂M. Then
  $$\int\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi\chi(M)$$
  where $\chi(M)$ is the Euler characteristic of M.

  So for our manifold we have: $$\int_{\partial M} k_g \, ds = 2\pi$$

  Now, k_g measures the curvature of ∂M at each point along the curve.

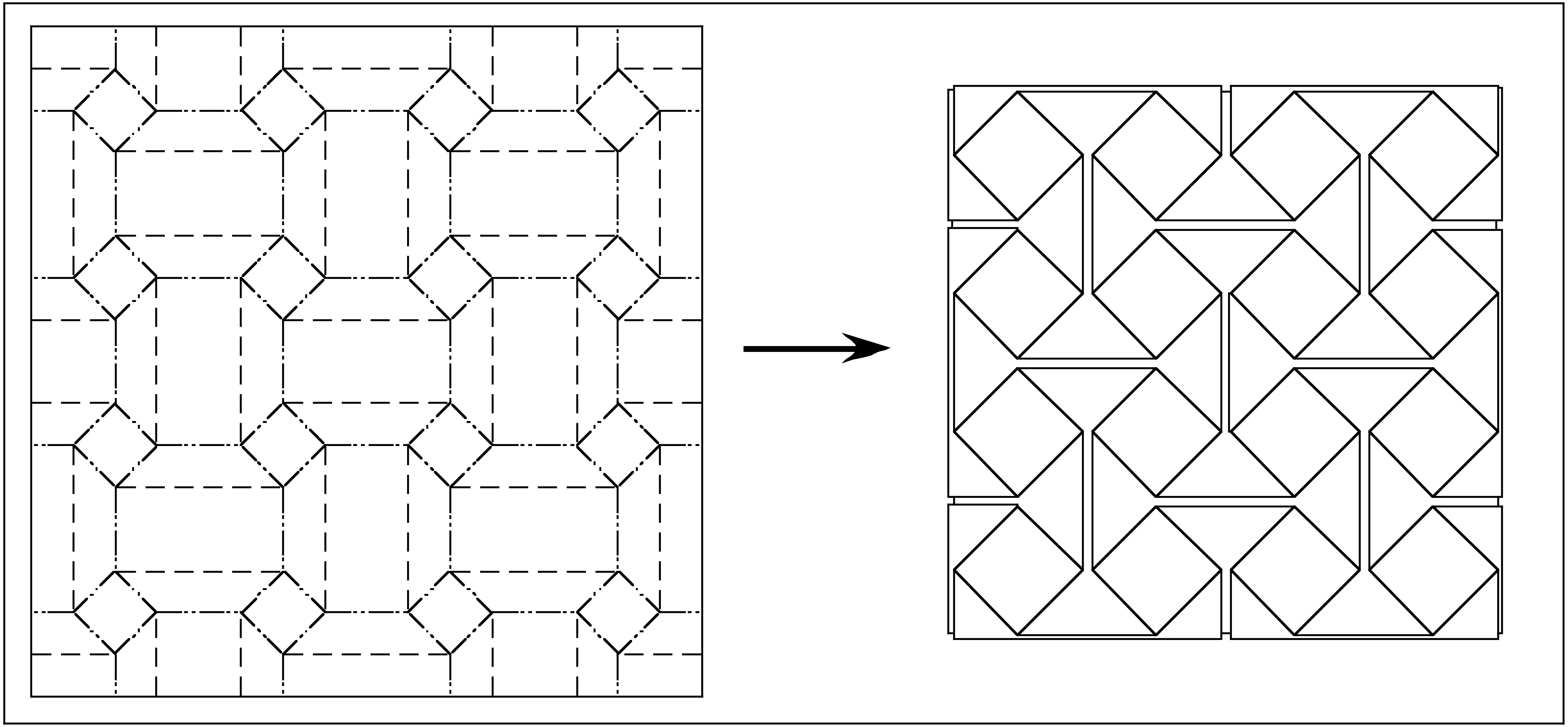
  Between two creases l_i and l_{i+1} we have $k_g = \alpha_i$ or $k_g = -\alpha_i$.

  At every mountain crease we have $k_g = -\pi$.

  At every valley crease we have $k_g = \pi$.

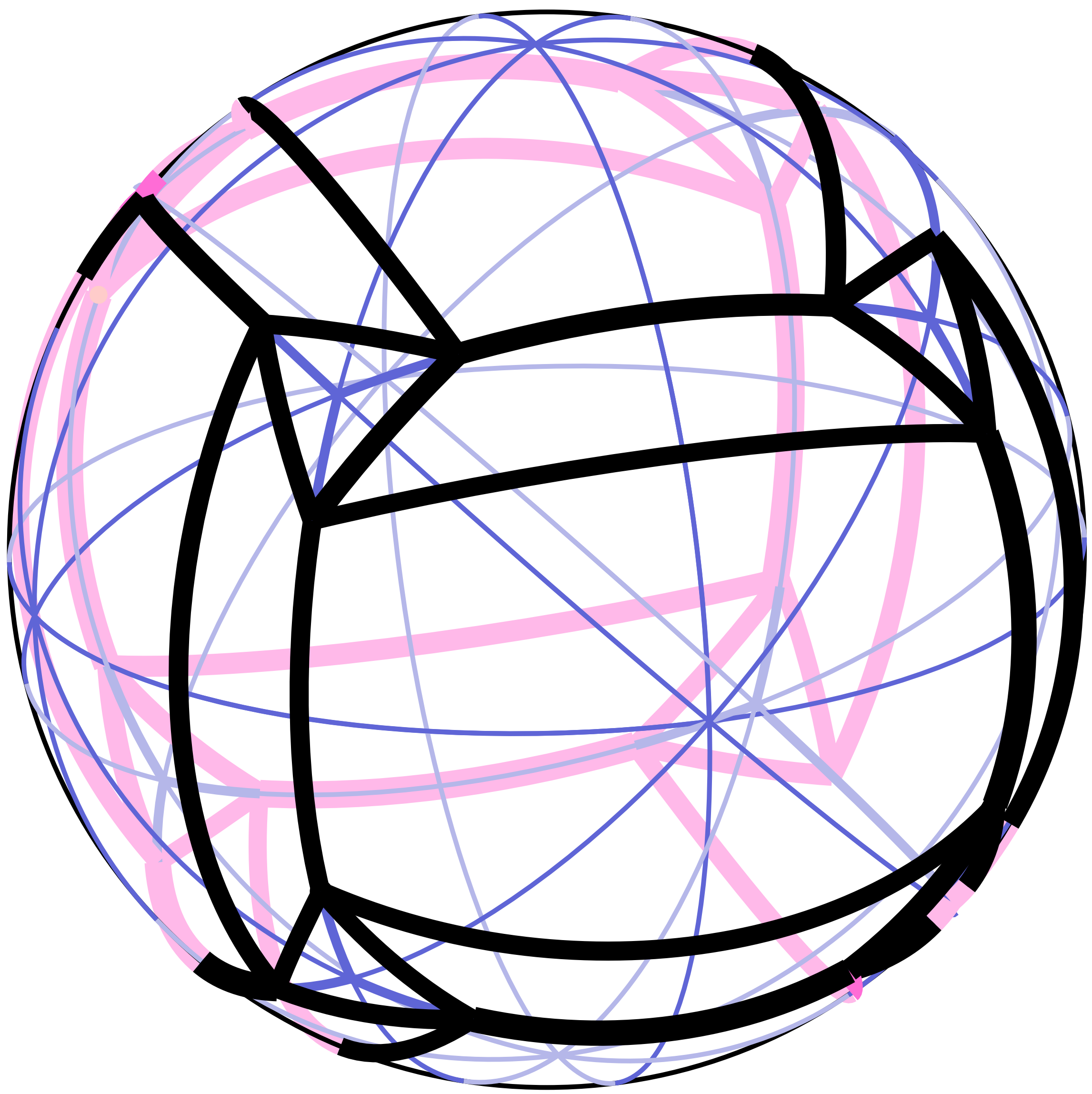
A weird idea I've been having...

- Origami tessellations have become more popular over the past 5 years.



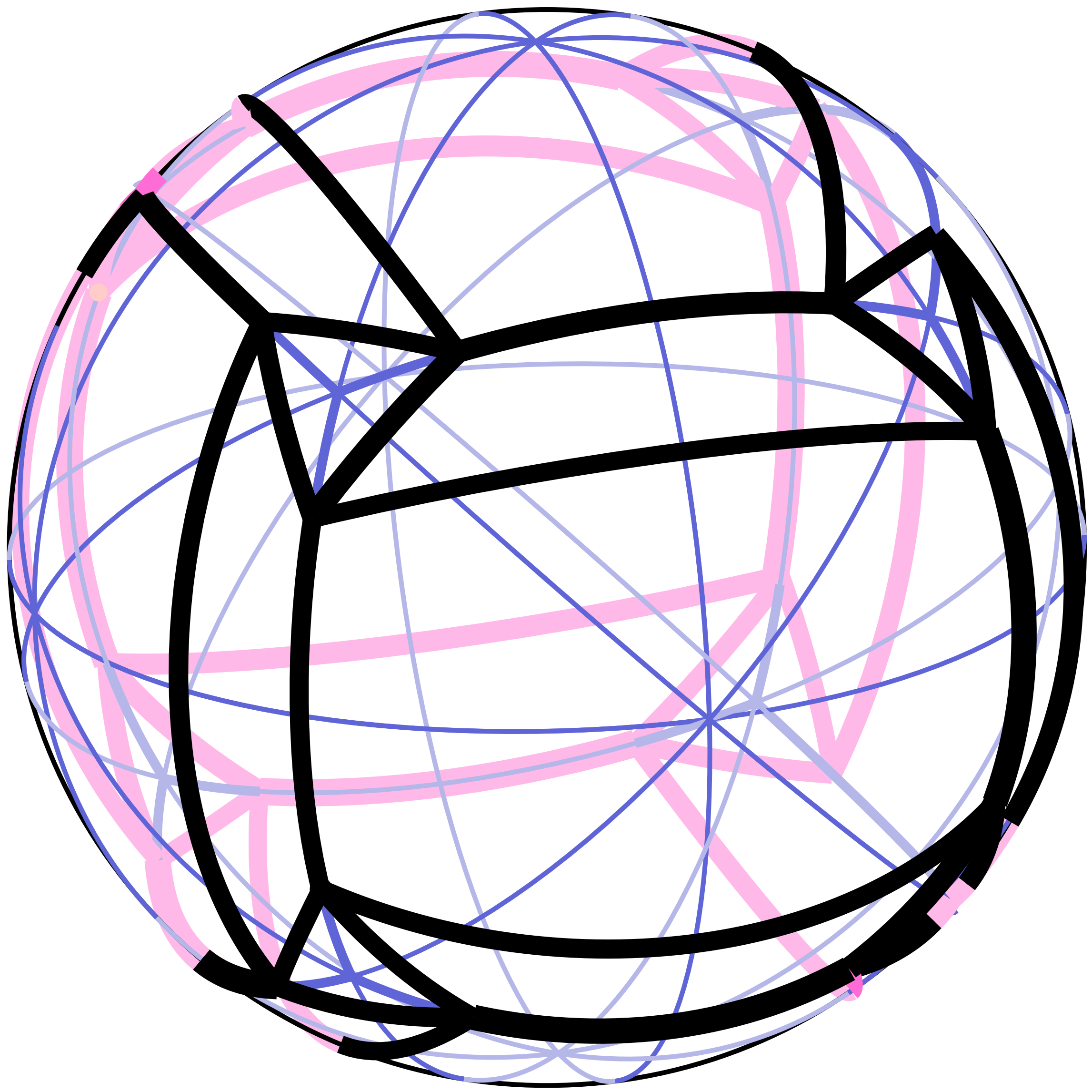
A weird idea I've been having...

- Could we make an origami tessellation on a sphere?



A weird idea I've been having...

- Could we make an origami tessellation on a sphere?



This is a lie!

A weird idea I've been having...

- Could we make an origami tessellation on a sphere?
- Let M be a sphere with a given radius. If $f : M \to M$ were an isometric folding whose crease pattern was a tessellation, then it would be a surjection, and would have $\deg f = \pm 1$. Robertson's Corollary 2 then tells us that $\Sigma(f) = \emptyset$, so this is impossible.

A weird idea I've been having...

- Could we make an origami tessellation on a sphere?

- Let M be a sphere with a given radius. If $f : M \to M$ were an isometric folding whose crease pattern was a tessellation, then it would be a surjection, and would have $\deg f = \pm 1$. Robertson's Corollary 2 then tells us that $\Sigma(f) = \emptyset$, so this is impossible.

- So what if f mapped to a sphere with a smaller radius?

A weird idea I've been having...

- Could we make an origami tessellation on a sphere?
- Let M be a sphere with a given radius. If $f : M \to M$ were an isometric folding whose crease pattern was a tessellation, then it would be a surjection, and would have $\deg f = \pm 1$. Robertson's Corollary 2 then tells us that $\Sigma(f) = \emptyset$, so this is impossible.
- So what if f mapped to a sphere with a smaller radius?
Impossible! You can't take a spherical polygon and put it onto a sphere with different radius and still have the sides be geodesics, and this would apply to any polygon of the crease pattern.

A weird idea I’ve been having...

- Could we make an origami tessellation on a sphere?
- What if our spherical paper was not the whole sphere, but just part of one, like a hemisphere?

A weird idea I've been having...

- Could we make an origami tessellation on a sphere?

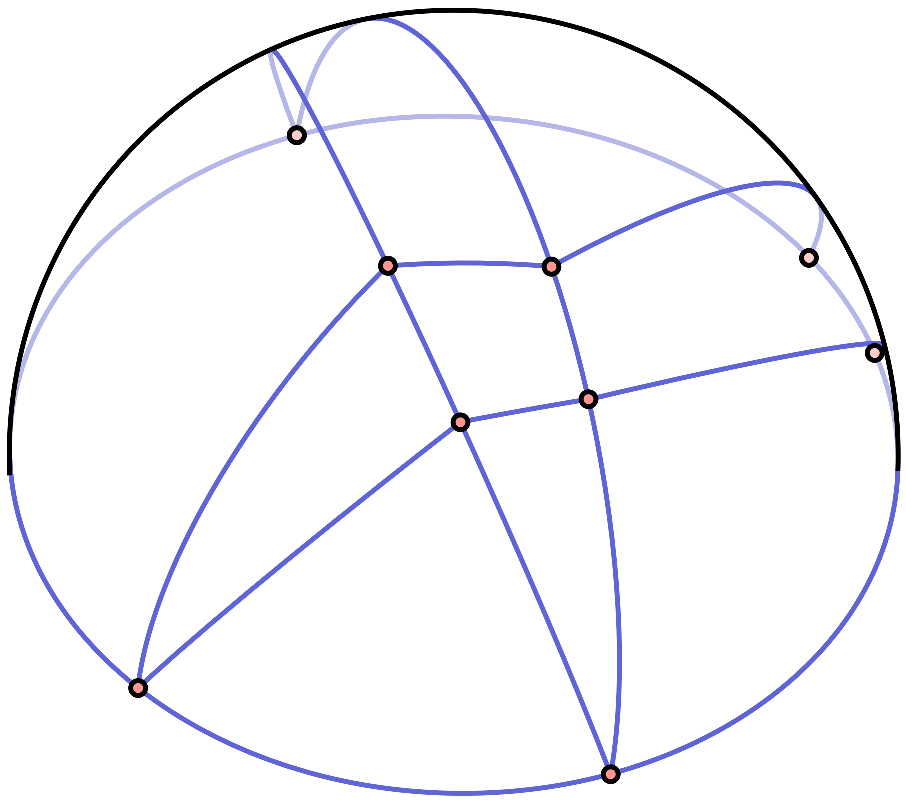
- What if our spherical paper was not the whole sphere, but just part of one, like a hemisphere?
  Now you're talking crazy. Any such crease pattern that would still qualify as an origami tessellation would have to tile the sphere. If the hemisphere could successfully fold, then so should the whole sphere!

  Note: This is not a rigorous argument! Perhaps the boundary of the hemisphere could absorb what's going wrong?

A weird idea I’ve been having...

- Could we make an origami tessellation on a sphere?

- What if our spherical paper was not the whole sphere, but just part of one, like a hemisphere?
  Still, I haven’t been able to make one yet.



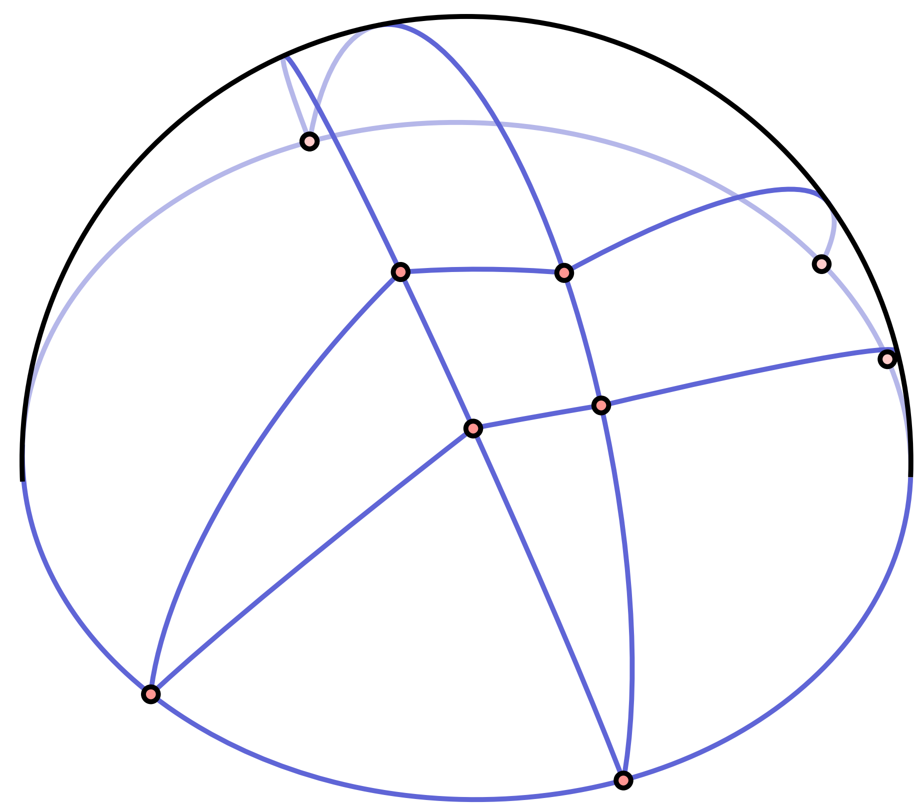
A weird idea I've been having...

- Could we make an origami tessellation on a sphere?

- What if our spherical paper was not the whole sphere, but just part of one, like a hemisphere?
  Still, I haven't been able to make one yet.

  Gauss-Bonnet, anyone?

$$\int\int_M K\ dA + \int_{\partial M} k_g\ ds = 2\pi\chi(M)$$



- What about origami tessellations on hyperbolic paper?
  Tom conjectures: No. But who knows?

Thank you!

thull@wnec.edu