Polyhedron (un)folding:

Folding: when can a polygon be glued along its boundary to form (exactly) a convex polyhedron? (only one layer allowed, unlike origami)

Unfolding: when can a polyhedral surface be cut & unfolded into one nonoverlapping planar piece?

Edge unfolding: just cut along polyhedron’s edges
General unfolding: can cut interior to faces

Summary:

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<th>edge unfolding</th>
<th>general unfolding</th>
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<tbody>
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<td>convex polyhedra</td>
<td>OPEN</td>
<td>ALWAYS</td>
</tr>
<tr>
<td>non convex polyhedra</td>
<td>NOT ALWAYS</td>
<td>OPEN</td>
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</table>

Big questions:

OPEN: does every convex polyhedron have an edge unfolding? [Dürer 1525; Shephard 1975]

OPEN: does every polyhedron without boundary have a general unfolding? [Bern, Demaine, Eppstein, Kuo 1999]
Curvature of a vertex = \(360^\circ - \sum\) incident face angles
- positive \(\Rightarrow\) convex cone
- zero \(\Rightarrow\) flat
- negative \(\Rightarrow\) saddle

Cutting = cuts in a valid unfolding
- only zero-curvature vertices can be flattened without cutting or local overlap
- any cutting spans all nonzero-curvature vertices
- indeed, if curvature \(< -k \cdot 360^\circ\)
  then cutting must have degree \(\geq k+1\)
- if polyhedron has no handles (sphere/disk, not torus)
  then cutting has no cycles (else \(>1\) piece)
- spanning forest
- connected component of cutting makes boundary component of unfolding
- if polyhedron has no boundary or handles & unfolding has no holes
  then cutting is a spanning tree
  \(-\) cf.:
  \(\text{boundary} \quad \Rightarrow \quad \text{hole}
  \)

- if polyhedron is convex
  then cutting is a spanning tree
Trivial bad example: [Bern, Demaine, Eppstein, Kuo 1999]

polyhedron with boundary & just one vertex, of negative curvature
- need \( \geq 2 \) cuts at vertex
- can't stop cutting until we reach the boundary (else could reglue cuts without change)

\( \Rightarrow \) disconnect surface
\( \Rightarrow \) no general unfolding

Shortest path between two points \( x \& y \) on polyhedron
- unfolds straight (geodesic)
- doesn't cross itself
- doesn't pass through a positive-curvature vertex
General unfoldings of convex polyhedra:

**Star of shortest paths from point** $x$ **to all other points**
- if two shortest paths touch beyond $x$
  then either one is a subpath of another
  or they touch only at their ends
  ($\Rightarrow$ nonunique shortest path)

**Cut locus / ridge tree** with respect to point $x$
- = points with nonunique shortest paths from $x$
- = Voronoi diagram of $x$
- = spanning tree of polyhedron
- leaves = the polyhedron vertices

**Source unfolding** [Sharir & Schorr 1986; Mount 1985; Mitchell, Mount, Papadimitriou 1987]
- cut along the cut locus
- unfold star of shortest paths from $x$
  $\Rightarrow$ star-shaped unfolding: boundary visible from $x$

**Star unfolding** [Alexandrov 1948; Aronov & O'Rourke 1992]
- cut along shortest paths from (generic) point $x$
  to every polyhedron vertex (star of cuts)
- much harder to prove nonoverlap
General unfoldings of convex polyhedra: (cont’d)

Extensions:
- Both source & star unfoldings generalize to $x = \text{geodesic path}$ [Itoh, O’Rourke, Vilcu 2008, 2009]
- Neither works for nonconvex polyhedra
- Source unfolding works in higher dimensions [Miller & Pak 2003]
- Source unfolding can be “continuously bloomed” without intersection [Demaine, Demaine, Hart, Iacono, Langerman, O’Rourke 2009/2010]
- Also, any unfolding can be refined to be continuously bloomable
- **OPEN**: True of star unfolding? all edge/general unfoldings? 
- **OPEN**: Other general unfoldings?
Edge-unfolding convex polyhedra:
- implicitly dates back to Albrecht Dürer's Painter's Manual [1525]
- possible for every example we've tried
  - e.g. Archimedean
  - heuristic/exhaustive search: commercial software, JavaView Unfold, Javagami, Unfold for Blender, Pepakura Designer]
  - Schlickenreider [1997] search
- all efficient algorithms we've tried fail [Schlickenreider 1997; Lucier 2006]
- some simple examples overlap
  e.g. sliver tetrahedron
- random cutting of random convex polyhedron overlaps with probability \( \to 1 \) as \( n \to \infty \) [Schevon & O’Kourke 1987]
hull of rand. pts. on sphere
- OPEN: prove this empirical observation

An approach: [Bern, Demaine, Rote, G. Price, ...]
- OPEN: edge-unfold convex terrains (project to a plane without intersection)
  \( \Rightarrow \) positive equilibrium stress
- OPEN: edge-unfold “almost flat” terrain/polyhedron (scale \( z \to \epsilon z \), \( \epsilon \to \) infinitesimal)
  \( \Rightarrow \) visible in plane
- challenging even for prisms/toids
  = convex hull of two parallel polygons
Edge-unfolding convex polyhedra: (cont’d)

Solved special classes:

- ≤ 6 vertices  [DiBiase 1990]
- pyramid = convex hull of convex polygon + point

- prism = convex hull of convex polygon + parallel offset

- prismoid = convex hull of two parallel convex polygons with matching angles  [O’Rourke 2001]

- dome = all faces share edge with single base  [O’Rourke - GFALOP]

- OPEN: prismatoids = convex hull of two parallel convex polygons
  - possible in “smooth” case  [Benbernou, Cahn, O’Rourke 2004]
  - band of side faces unfolds  [Aloupis 2005]
Fewest nets: edge-unfold convex polyhedron into a "small" number of pieces
- want to know whether 1 is possible
- $F = \#\text{ faces}$ is trivial (cut out each)
- $\frac{2}{3}F$ by pairing together $\frac{2}{3}$ of faces [Spriggs 2003]
- $\frac{1}{2}F$ by fancier argument [Dujmović, Morin, Wood 2004]
- better bounds [Pinciu 2007]
- **OPEN**: $o(F)$ possible?

Edge-unfolding nonconvex polyhedra:
- trivial unfoldable example: insufficient area in donut hole [Biedl et al. 1999]
- with all faces $\sim$ disks: can't connect two X's [Biedl et al. 1999]
- with all faces triangles $\Rightarrow$ two share only one edge ("topologically convex") [Bern, Demaine, Eppstein, Kuo, Mantler, Snoeyink 2003]
- smaller examples [Grüenbaum]
Triangulated unfoldable example:

- Suppose base vertices of spike have neg. curvature, even without one spike $\Delta$
  (brim angle $= 300^\circ - \varepsilon$,
  spike angle $= 90^\circ - \varepsilon \Rightarrow 390^\circ - 2\varepsilon$)
- Claim: Can't edge-unfold a hat by itself
  - Spanning forest has $\geq 2$ leaves
  - Can't be at negative curvature vertices
  - Can't have two on boundary
  - One at peak, one on boundary
  - Two possibilities remain
  - Both leave all but one spike $\Delta$
     at a base vertex of spike
  $\Rightarrow$ Must be a path of cuts
  between two boundary vertices, interior to hat
  - These 4 paths force cycle on 4 vertices

$\Rightarrow$ No one-piece edge unfolding $\Box$